



Aalborg Universitet

AALBORG UNIVERSITY  
DENMARK

## Invariants of directed spaces

Raussen, Martin

*Publication date:*  
2006

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*  
Raussen, M. (2006). *Invariants of directed spaces*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2006-28

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal -

### Take down policy

If you believe that this document breaches copyright please contact us at [vbn@aub.aau.dk](mailto:vbn@aub.aau.dk) providing details, and we will remove access to the work immediately and investigate your claim.

**Invariants of directed spaces**

by

Martin Raussen

R-2006-28

August 2006

DEPARTMENT OF MATHEMATICAL SCIENCES  
AALBORG UNIVERSITY

Fredrik Bajers Vej 7 G ■ DK-9220 Aalborg Øst ■ Denmark

Phone: +45 96 35 80 80 ■ Telefax: +45 98 15 81 29

URL: [www.math.auc.dk/research/reports/reports.htm](http://www.math.auc.dk/research/reports/reports.htm)



# INVARIANTS OF DIRECTED SPACES

MARTIN RAUSSEN

## 1. INTRODUCTION

**1.1. Background.** With motivations arising originally from concurrency theory within Computer Science, a new field of research, directed algebraic topology, has emerged. The main characteristic is, that it involves spaces of “directed paths” (or timed paths, executions): these directed paths can be concatenated, but in general *not* reversed; time is not reversible. These executions can be viewed as objects themselves (this is the point of view of Gaucher, c.f., e.g. [9, 10]) or as elements of subspaces of spaces of paths in an underlying topological space (with suitable concatenations). We will apply the latter approach, compare also [14, 7].

Examples of “directed paths” occur in spaces with a (local) partial order [8] motivated from Higher Dimensional Automata (HDA) models in concurrency; for these consult e.g. the recent [20]. Directed paths show the same behaviour (calculations along them yield the same result) if they are *dihomotopic*, i.e., if they can be connected through a one-parameter family of directed paths. Since one cannot reverse directed paths, one can no longer expect invariants in “reversible structures” like (fundamental) groups etc.; they will rather live in categories (like the fundamental category and others discussed further on in this article).

A nice and flexible framework for directed paths was introduced by Marco Grandis with the notion of  $d$ -spaces and, in particular, of  $d$ -paths (cf. Def. 2.1) on a topological space  $X$ . In [14] and the subsequent [13], he developed a framework for directed homotopy theory. In particular, he proved a van-Kampen theorem for fundamental categories that allows to do calculations on the (refined)  $d$ -homotopy sets (cf. Def. 2.1) of directed paths. Moreover, in these and more recent papers by Grandis, the methodology was extended to directed homotopy on general categories.

The objects of a fundamental category consist of all its points; this category is therefore a very large tool and not discretized, as is the case with the fundamental group of a topological space. It was the aim of [7], to divide the underlying space into “components” in a systematic way, so that one only needs to investigate the (dihomotopy classes of) directed paths between components. The outcome was the *component category* of the fundamental category of a space, with certain generalizations to component categories of other categories. The approach chosen in [7] had the drawback, that it

---

The author thanks Institut Mittag-Leffler, Djursholm, Sweden, for its hospitality during his participation in its Algebraic Topology programme in April 2006, that allowed him to organise the material in this article.

involves *choices* of certain subcategories and thus does not yield *the* category of components. In [11], the authors overcome this dilemma by an additional pullback/pushout requirement (cf. Def. 4.3) giving rise to unique well-defined component categories. On the other hand, it is not always clear, how natural this extra requirement is. In particular, in the presence of loops, this method (and also those of the previous [7]) does not give satisfying results.

**1.2. Aims of this paper.** This paper contains several contributions that are linked to each other. First of all, we begin a systematic study of the topology of spaces of directed paths and not only of their path components (which are studied via the fundamental category). To get organized, we need to introduce a zoo of indexing categories, most prominently the so-called *preorder category* of a directed space, cf. Section 3.2: Its objects are *pairs* of points, reflecting the fact that a directed space is mainly characterized by the directed paths between a given *source* and a given *target* (and the relations between those). The indexing categories come along with functors to well-known categories such as the (homotopy) category of topological spaces, homology functors to abelian groups and so on. Morphisms in the indexing category are then regarded as (weakly) invertible if they are mapped into *isomorphisms* by the functor. As a result, one can apply the localization method leading to components suggested in [7] and modified in [11] *to the indexing category*.

Other possibilities of compressing the size of categories and obtaining minor (or even minimal) models of categories and retaining the essential “directed information” were pursued in more recent work of M. Grandis[16, 15] and of S. Krishnan[17]. It is certainly desirable to compare the various methods and their results in more detail.

In ordinary algebraic topology, topological spaces are considered to have the same shape if they are homotopy equivalent. It is not clear what the corresponding notion ought to be for directed spaces, and the obvious generalization has very weak properties with respect to path spaces: corresponding spaces of directed paths are in general not homotopy equivalent to each other. The “missing link” is formalized by the notion of an automorphic homotopy flow (Section 5) on a given directed space; such a homotopy flow produces a family of directed flow lines, but it need not give rise to self-homeomorphisms. These automorphic homotopy flows can usually only flow within limited regions; these regions can then be used to give an alternative way to obtain components and component categories.

Components and component categories are usually not at all preserved under directed continuous maps. We investigate under which conditions at least a directed homotopy equivalence gives rise to equivalent component categories. If a computation of the component categories of two spaces yields inequivalent results, then they cannot be directed homotopy equivalent (under an additional coherence condition).

**1.3. Outline of the paper.** In Section 2 and 3, we extend the toolbox associated to a directed space from a single tool (the fundamental category) to a kit consisting of several related categories and functors reflecting both the underlying path spaces and their (higher) homotopy and homology. In order to make concatenation associative, we use

the notion of directed traces (directed paths up to directed reparametrization) investigated in [5]. Path spaces are *dipointed* (source and target); this is reflected in the (indexing) preorder category of a directed space, on which we can define the most important functors (organising the path spaces, their homology or homotopy groups etc.). In particular, we “reorganize” the information contained in the fundamental category through a functor from the preorder category to *Sets* as a special case.

Section 4 studies methods of inverting (a collection of) morphisms in the preorder category that induce isomorphisms under the functor under investigation. The preorder setting allows, e.g., to invert a directed path from  $x$  to  $y$  seen as a morphism between certain pairs, but not between others; this is quintessential for spaces with loops. Systematic inversion requires certain extension properties which allow to arrive at component *categories*.

In Section 5, we introduce automorphic homotopy flows on a given directed space and investigate their main properties. This is first applied in Section 6 to the definition of a dihomotopy equivalence between two given directed spaces; these dihomotopy equivalences are then shown to induce homotopy equivalences on related spaces of directed paths.

In Section 7, we describe an alternative way to arrive at components in an organized way by exploiting automorphic homotopy flows. These homotopy flows lead to invertibles in an extended preorder category that satisfy the extension properties right away. We show in Section 8, that the component categories from this approach are equivalent under dihomotopy equivalences satisfying a coherence condition. Finally, we sketch how these ideas might be generalized from functors on preorder categories to functors on more general categories.

## 2. REPARAMETRIZATIONS AND TRACES

**2.1. Review on  $d$ -spaces.** We start with a review of the basic notions, mainly taken from [14]: Let  $X$  denote a Hausdorff topological space, let  $P(X) = C(I, X) = X^I$  denote the space of all paths in  $X$ , i.e., of all continuous maps from the unit interval  $I$  into  $X$  equipped with the compact-open topology.

**Definition 2.1.** [14]

- (1) A  $d$ -space is a topological space  $X$  together with a set  $\vec{P}(X) \subseteq P(X)$  of continuous paths  $I \rightarrow X$  such that
  - (a)  $\vec{P}(X)$  contains all constant paths;
  - (b)  $\vec{P}(X)$  is closed under concatenation;
  - (c)  $p \circ \varphi \in \vec{P}(X)$  for any  $p \in \vec{P}(X)$  and any continuous increasing (not necessarily surjective, not necessarily strictly increasing) map  $\varphi : I \rightarrow I$ ;
- (2) Elements of  $\vec{P}(X)$  are called  $d$ -paths. For  $x, y \in X$ ,  $\vec{P}(X)(x, y)$  consists of all  $d$ -paths  $p \in \vec{P}(X)$  with source  $x$  and target  $y$  ( $p(0) = x, p(1) = y$ ).
- (3) A continuous map  $f : X \rightarrow Y$  between two  $d$ -spaces is called a  $d$ -map if  $f(\vec{P}(X)) \subseteq \vec{P}(Y)$ .

For the *oriented* unit interval  $\vec{I}$ ,  $\vec{P}(\vec{I})$  consists of the continuous increasing maps  $\varphi : I \rightarrow I$ . A  $d$ -path on  $X$  is then a  $d$ -map from  $\vec{I}$  to  $X$ .

The product of two  $d$ -spaces is a  $d$ -space in a natural way. For homotopy purposes, we will in particular be interested in the products  $X \times I$  and  $X \times \vec{I}$  of a  $d$ -space  $X$  with the unoriented interval  $I$  ( $\vec{P}(I) = P(I)$ ) and the oriented interval  $\vec{I}$ .

- Definition 2.2.** (1) A dihomotopy (between  $d$ -maps  $f = H_0, g = H_1 : X \rightarrow Y$ ) is a  $d$ -map  $H : X \times I \rightarrow Y$  (i.e., each map  $H_t$  is a  $d$ -map);
- (2) A  $d$ -homotopy  $f \xrightarrow{H} g$  (from  $f = H_0$  to  $g = H_1$ ) in  $X$  is a  $d$ -map  $H : X \times \vec{I} \rightarrow Y$  (i.e., additionally all paths  $H(x, t), x \in X$  are  $d$ -paths in  $Y$ );
- (3) Dihomotopy is the equivalence relation defined by (1);  $d$ -homotopy is the equivalence relation generated by (2) (as transitive and symmetric closure).

Obviously,  $d$ -homotopic maps are dihomotopic; the opposite is in general not true. It is true, though, for paths in certain cubical complexes [6].

**2.2. Reparametrization equivalence.** This section reviews the main definitions and results from [5] concerning continuous *reparametrizations* of  $d$ -paths. A *reparametrization*  $\varphi : I \rightarrow I$  is a *continuous surjective increasing* self-map of the interval  $I$ ; in particular:  $s \leq t \Rightarrow \varphi(s) \leq \varphi(t)$  and  $\varphi(0) = 0, \varphi(1) = 1$ . A *regular reparametrization*  $\varphi : I \rightarrow I$  is a reparametrization satisfying  $s < t \Rightarrow \varphi(s) < \varphi(t)$ ; in other words, it is a self-homeomorphism of the interval respecting end-points.

Let  $\text{Rep}_+(I)$  denote the *topological monoid* of all reparametrizations. A topology on  $\text{Rep}_+(I)$  is induced from the compact-open topology on the space  $C(I, I)$  (inherited from the supremum metric) of all self-maps of the interval. Let  $\text{Homeo}_+(I) \subset \text{Rep}_+(I)$  denote the *topological group* of all regular reparametrizations.

The monoid  $\text{Rep}_+(I)$  acts, for all  $x, y \in X$ , continuously on  $\vec{P}(X)(x, y)$  by composition (on the right). Remark that this action preserves  $d$ - and di-homotopy classes.

**Definition 2.3.** [5] Two  $d$ -paths  $p, q \in \vec{P}(X)$  are called *reparametrization equivalent* if there exist reparametrizations  $\varphi, \psi$  such that  $p \circ \varphi = q \circ \psi$ .

**Proposition 2.4.** [5] *Reparametrization equivalence is an equivalence relation, in particular it is transitive.*

Taking quotients with respect to reparametrization equivalence yields a quotient map  $q_P : \vec{P}(X)(x, y) \rightarrow \vec{T}_P(X)(x, y) := \vec{P}(X)(x, y) / \simeq$ ; we endow this quotient space with the quotient topology.

**Definition 2.5.** [5]

- (1) A  $d$ -path  $\alpha : I \rightarrow X$  is called *regular* if it is either constant or if there is no non-trivial subinterval  $J \subset I$  on which it is constant.
- (2) Let  $\vec{R}(X)(x, y) \subset \vec{P}(X)(x, y)$  denote the space of all regular  $d$ -paths from  $x$  to  $y$  with the induced topology.

**Proposition 2.6.** [5] *Let  $x \neq y$  be elements of a  $d$ -space  $X$ . The action of  $\text{Homeo}_+(I)$  on  $\vec{R}(X)(x, y)$  is free.*

The group action of  $\text{Homeo}_+(I)$  defines an (orbit) equivalence relation  $\simeq$  and a quotient map  $q_R : \vec{R}(X)(x, y) \rightarrow \vec{T}_R(X)(x, y) := \vec{P}(X)(x, y) / \simeq$ . We arrive at a commutative diagram of inclusions and quotient spaces

$$\begin{array}{ccc} \vec{P}(X)(x, y) & \hookrightarrow & \vec{R}(X)(x, y) \\ q_P \downarrow & & \downarrow q_R \\ \vec{T}_P(X)(x, y) & \xrightarrow{i} & \vec{T}_R(X)(x, y). \end{array}$$

**Proposition 2.7.** [5] *The map  $i : \vec{T}_P(X)(x, y) \rightarrow \vec{T}_R(X)(x, y)$  is a homeomorphism.*

The proof in [5] proceeds in three steps: First, we show that every  $d$ -path is a reparametrization of a regular  $d$ -path. This yields surjectivity. Injectivity relies on a factorization property for reparametrizations. It is obvious that  $i$  is continuous. To see that it is also open relies on the fact that  $\text{Homeo}_+(I)$  is dense in  $\text{Rep}_+(I)$ , cf. [5].

**Lemma 2.8.** [5] *For  $x \neq y$ , the quotient map  $q_R : \vec{R}(X)(x, y) \rightarrow \vec{T}_R(X)(x, y)$  is a weak homotopy equivalence.*

*Proof.* The free group action yields a fibration with contractible fiber  $\text{Homeo}_+(I)$ .  $\square$

**Remark 2.9.** It would be interesting to know whether (or under which conditions) the quotient map  $q_R$  from Lemma 2.8 is a genuine homotopy equivalence.

In conclusion, for calculations of homotopy or homology invariants, we can use any of the spaces  $\vec{R}(X)(x, y)$ ,  $\vec{T}_P(X)(x, y)$  or  $\vec{T}_R(X)(x, y)$ . In many cases, the conclusion of Lemma 2.8 holds also for  $x = y$ ; cf. [5]. In the following, we will use the notation  $\vec{T}(X)(x, y)$  for both  $\vec{T}_P(X)(x, y)$  and  $\vec{T}_R(X)(x, y)$ .

### 3. THE TRACE CATEGORY AND ITS RELATIVES

In this section, we describe various indexing categories that can be used to organize the spaces of traces with given source and target. One may note a certain analogy to various categories (i.e., orbit categories) organising  $G$ -spaces in equivariant topology, cf. e.g. [19].

**3.1. The trace category.** The *trace category*  $\vec{T}(X)$  of a  $d$ -space has the elements of  $X$  as objects; the morphisms from  $x$  to  $y$  are given by  $\vec{T}(X)(x, y)$  – with the topology as a quotient space of  $\vec{R}(X)(x, y)$ . Composition on  $\vec{T}(X)$  is inherited from concatenation on  $\vec{R}(X)$ . The latter is only associative up to reparametrization, just enough to make composition on  $\vec{T}(X)$  associative!

A  $d$ -map  $f : X \rightarrow Y$  between two  $d$ -spaces  $X$  and  $Y$  induces a functor  $\vec{T}(f) : \vec{T}(X) \rightarrow \vec{T}(Y)$  by composition on the left on morphisms, i.e.,  $\vec{T}(f) : \vec{T}(X)(x, y) \rightarrow \vec{T}(Y)(fx, fy)$  is given by  $\vec{T}(f)[\psi] = [f \circ \psi]$ .

The *fundamental category*  $\vec{\pi}_1(X)$  [18, 7] arises from the trace category  $\vec{T}(X)$  as the category of path components, with the *dihomotopy* relation. Concatenation on the trace spaces is homotopy invariant and factors over the fundamental category. In particular, left and right concatenation define maps

$$(3.1) \quad C_l : \vec{\pi}_1(X)(x, y) \rightarrow \text{Mor}(\vec{T}(X)(y, z), \vec{T}(X)(x, z)) \text{ and}$$

$$(3.2) \quad C_r : \vec{\pi}_1(X)(y, z) \rightarrow \text{Mor}(\vec{T}(X)(x, y), \vec{T}(X)(x, z))$$

with morphisms in the homotopy category  $Ho - Top$ .

*Remark 3.1.* The fundamental category in the sense of Grandis [14] is different, since a  $d$ -homotopy between  $d$ -paths is not just a path in the space of  $d$ -paths – but for our considerations, everything works if one uses the  $d$ -homotopy relation instead of dihomotopy.

An algebraic counterpart is the homology category  $\vec{H}(X)$  with points as objects and with  $\vec{H}_*(X)(x, y) = \bigoplus_{n \geq 0} H_n(\vec{T}(X)(x, y); R)$ , the *total* homology with coefficients in a ring  $R$ . A composition law is then given by a generalization of the Pontryagin-product for  $H$ -spaces, i.e.,

$$H_*(\vec{T}(X)(x, y)) \times H_*(\vec{T}(X)(y, z)) \xrightarrow{\times} H_*(\vec{T}(X)(x, y) \times \vec{T}(X)(y, z)) \xrightarrow{*} H_*(\vec{T}(X)(x, z)),$$

where the first map is given by the homological cross-product and the second is induced by concatenation.

Taking homology of the trace spaces corresponds to functors  $\vec{H}_*(X) : \vec{T}(X) \rightarrow \vec{H}_*(X)$ . A  $d$ -map  $f : X \rightarrow Y$  between two  $d$ -spaces  $X$  and  $Y$  induces natural transformations  $f_* : \vec{H}_*(X) \rightarrow \vec{H}_*(Y)$  – with group homomorphisms between the morphism groups.

A similar construction can be done for homotopy groups. Define homotopy categories  $\vec{\Pi}_{n+1}(X)$  with points of  $X$  as objects and with

$$\vec{\Pi}_{n+1}(X)(x, y) := \coprod_{\sigma \in \vec{T}(X)(x, y)} \vec{\pi}_n(\vec{T}(X)(x, y); \sigma) \text{ with composition law given by}$$

$$\vec{\pi}_n(\vec{T}(X)(x, y); \sigma_1) \times \vec{\pi}_n(\vec{T}(X)(y, z); \sigma_2) \xrightarrow{\cong} \vec{\pi}_n(\vec{T}(X)(x, y) \times \vec{T}(X)(y, z); (\sigma_1, \sigma_2)) \xrightarrow{*} \vec{\pi}_n(\vec{T}(X)(x, z); \sigma_1 * \sigma_2);$$

the second map is again induced by concatenation. Since homotopy groups – up to isomorphism – only depend on the connected component of the base point, one can instead index the coproduct by  $\vec{\pi}_1(X)(x, y)$ .

**3.2. Preorder categories and homology.** More useful indexing devices are several variants of *preorder categories*  $\vec{D}(X)$  of a  $d$ -space  $X$ . They have all the same objects, but different morphisms. A  $d$ -space  $X$  comes equipped with a natural preorder  $x \preceq y \Leftrightarrow \vec{P}(X)(x, y) \neq \emptyset$ . In all preorder categories, the objects are the pairs  $(x, y) \in X \times X$  with  $x \preceq y$ .



The morphisms in  $\vec{D}(X)$  are  $\vec{D}(X)((x, y), (x', y')) := \vec{T}(X)(x', x) \times \vec{T}(X)(y, y')$  with composition given by pairwise contra-, resp. covariant concatenation. Remark that every morphism  $(\sigma_x, \sigma_y) \in \vec{T}(X)(x', x) \times \vec{T}(X)(y, y')$  decomposes as follows:  $(\sigma_x, \sigma_y) = (c_{x'}, \sigma_y) \circ (\sigma_x, c_y) = (\sigma_x, c_{y'}) \circ (c_x, \sigma_y)$  with  $c_u \in \vec{T}(X)(u, u)$  the constant trace at  $u \in X$ .

A  $d$ -map  $f : X \rightarrow Y$  induces a functor  $\vec{D}(f) : \vec{D}(X) \rightarrow \vec{D}(Y)$  with  $\vec{D}(f)(x, y) = (fx, fy)$  and  $\vec{D}(f)(\sigma_x, \sigma_y) = (\vec{T}(f)(\sigma_x), \vec{T}(f)(\sigma_y)) = (f \circ \sigma_x, f \circ \sigma_y)$ .

Trace spaces can be organised by the trace space functor  $\vec{T}^X : \vec{D}(X) \rightarrow \text{Top}$  given by  $\vec{T}^X(x, y) = \vec{T}(X)(x, y)$  and  $\vec{T}^X(\sigma_x, \sigma_y)(\sigma) := \sigma_x \circ \sigma \circ \sigma_y \in \vec{T}(X)(x', y')$  for  $\sigma \in \vec{T}(X)(x, y)$ . A  $d$ -map  $f : X \rightarrow Y$  induces a natural transformation  $\vec{T}(f)$  from  $\vec{T}^X$  to  $\vec{T}^Y$ .

A "smaller" homotopical variant is given by the category  $\vec{D}_\pi(X)$  with the same objects as above and with  $\vec{D}_\pi(X)((x, y), (x', y')) := \vec{\pi}_1(X)(x', x) \times \vec{\pi}_1(X)(y, y')$ . It comes with a functor  $\vec{T}_\pi^X : \vec{D}_\pi(X) \rightarrow \text{Ho} - \text{Top}$  into the homotopy category; a  $d$ -map  $f : X \rightarrow Y$  induces a natural transformation  $\vec{T}_\pi(f)$  from  $\vec{T}_\pi^X$  to  $\vec{T}_\pi^Y$ . Together with the (vertical) forgetful functors, we obtain a commutative diagram

$$(3.3) \quad \begin{array}{ccc} \vec{D}(X) & \xrightarrow{\vec{T}^X} & \text{Top} \\ \downarrow & & \downarrow \\ \vec{D}_\pi(X) & \xrightarrow{\vec{T}_\pi^X} & \text{Ho} - \text{Top}. \end{array}$$

The functors  $\vec{T}_X$  and  $\vec{T}_\pi^X$  may be composed with homology functors into categories of (graded) abelian groups,  $R$ -modules or graded rings. In particular, we obtain, for  $n \geq 0$ , functors  $\vec{H}_{n+1}(X) : \vec{D}(X) \rightarrow \text{Ab}$  with  $(x, y) \mapsto H_n(\vec{T}(X)(x, y))$  and  $(\sigma_x, \sigma_y)_*$  given by the map induced on  $n$ -th homology groups by concatenation with those two traces on trace space  $\vec{T}(X)(x, y)$ . This functor factors obviously over  $\vec{D}_\pi(X)$ . In the same spirit, one can define homology with coefficients and cohomology; a  $d$ -map  $f : X \rightarrow Y$  induces a natural transformation  $\vec{H}_{n+1}(f) : \vec{H}_{n+1}(X) \rightarrow \vec{H}_{n+1}(Y), n \geq 0$ .

*Remark 3.2.* It would be nice to have better theoretical and computational tools concerning the homology of trace (or path) spaces. First steps in this direction will be the subject of a current master's thesis.

**3.2.1. Preorder endomorphism categories.** The effect of self- $d$ -maps  $f : X \rightarrow X$  (from a  $d$ -space  $X$  to itself) on the trace spaces is reflected by the preorder endomorphism category  $\vec{DE}(X)$ : It has the same objects as the category  $\vec{D}(X)$ , whereas  $\vec{DE}(X)(x, y)(x', y') := \{f : X \rightarrow X \mid f \text{ a dimap with } f(x) = x', f(y) = y'\}$  and composition is composition of  $d$ -maps. This category organises trace spaces through the functor  $\vec{T}^{EX} : \vec{DE}(X) \rightarrow \text{Top}$  with  $\vec{T}^{EX}(x, y) := \vec{T}(X)(x, y)$  and  $\vec{T}^{EX}(f) := \vec{T}(f) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(x', y')$  for  $f \in \vec{DE}(X)(x, y)(x', y')$ .

The quotient functor into  $Ho - Top$  factors over a category, where the morphisms are homotopy classes of  $d$ -maps with homotopies fixing “end points”. Also these functors can be composed with homology functors. Remark that these preorder endomorphism categories are *not* functorial with respect to  $d$ -maps between different  $d$ -spaces. We do not go into details, since it seems to be difficult to obtain localizations (cf. Section 7) with good properties within these categories.

It is sometimes more significant to restrict the preorder endomorphism category to one with the same objects, but with fewer endomorphisms:  $\vec{D}E_0(X)(x, y)(x', y') := \{f : X \rightarrow X \mid f \text{ a dimap } d - \text{homotopic to } id_X \text{ with } f(x) = x', f(y) = y'\}$ .

**3.3. Factorization categories and higher homotopy.** For indexing purposes, that are finer than those in Sect. 3.2, it is convenient to consider the *factorization category*  $F\vec{T}(X)[1]$  of the trace category  $\vec{T}(X)$ : The objects of  $F\vec{T}(X)$  are just the morphisms of the trace category  $\vec{T}(X)$ . Moreover, we define

$$F\vec{T}(X)(\sigma_{xy}, \sigma'_{x'y'}) := \{(\varphi_{x'x}, \varphi_{yy'}) \in \vec{T}(X)(x', x) \times \vec{T}(X)(y, y') \mid \sigma'_{x'y'} = \varphi_{yy'} \circ \sigma_{xy} \circ \varphi_{x'x}\};$$

composition is defined as in Sect. 3.2 above, by restriction.

This category comes with a functor  $F\vec{T}^X : F\vec{T}(X) \rightarrow Top_*$  into the category of *pointed* topological spaces. It associates to  $\sigma_{xy}$  the pointed topological space  $(\vec{T}(X)(x, y); \sigma_{xy})$ . As in Sect. 3.2, one may consider homotopical variants, a category  $F_\pi \vec{T}(X)$  with pairs of *dihomotopy classes* of commuting traces as morphisms and a functor  $F_\pi \vec{T}^X : F_\pi \vec{T}(X) \rightarrow Ho - Top_*$ .

The main interest in these categories and functors arises after composition with homotopy functors from either  $Top_*$  or from  $Ho - Top_*$  and to arrive at homotopy functors  $\vec{\pi}_2(X) : F\vec{T}(X) \rightarrow Groups$  and  $\vec{\pi}_{n+1}(X)(\sigma_{xy}) : F\vec{T}(X) \rightarrow Ab$ ,  $n > 1$ , given by  $\vec{\pi}_{n+1}(X) := \pi_n(\vec{T}(X)(x, y); \sigma_{xy})$  and the obvious induced maps. These homotopy functors factor over  $F_\pi \vec{T}(X)$ . The constructions are again functorial with respect to  $d$ -maps.

A dihomotopy between dipaths  $\sigma_1$  and  $\sigma_2$  corresponds to a change of base point in the same component of the trace space. In particular, the associated homotopy groups  $\vec{\pi}_{n+1}(X)(\sigma_1)$  and  $\vec{\pi}_{n+1}(X)(\sigma_2)$  are isomorphic. Another way to express this fact: Up to isomorphism, the functor  $\vec{\pi}_{n+1}(X)$  factorizes over the factorization category  $F\vec{\pi}_1(X)$  of the fundamental category  $\vec{\pi}_1(X)$ .

**3.3.1. Endomorphism trace categories.** Looking at the effect of  $d$ -self maps leads to the consideration of yet another category related to a  $d$ -space  $X$ . The endomorphism trace category  $E\vec{T}(X)$  has the same objects as  $F\vec{T}(X)$ , whereas  $E\vec{T}(X)(\sigma_{xy}, \sigma'_{x'y'}) := \{f : X \rightarrow X \mid f \text{ a } d\text{-map with } f(\sigma_{xy}) = \sigma'_{x'y'}\}$ ; equality means of course reparametrization equivalence of representatives. Endomorphism trace categories are not functorial with respect to  $d$ -maps between different  $d$ -spaces; they do not seem to enjoy good enough properties for localization purposes (cf. Sect. 7).

The significance of endomorphism trace categories arises from the functor  $E\vec{T}^X : E\vec{T}(X) \rightarrow Top_*$ , which to a  $d$ -map  $f$  seen as morphism from  $\sigma_{xy}$  to  $\sigma'_{x'y'}$  associates the pointed map  $\vec{T}(f) : (\vec{T}(X)(x, y); \sigma_{xy}) \rightarrow (\vec{T}(X)(x', y'); \sigma'_{x'y'})$ . Also this functor can be composed with homotopy functors.

As before, one may restrict attention to the category  $E_0\vec{T}(X)$  with  $E_0\vec{T}(X)(\sigma_{xy}, \sigma'_{x'y'}) := \{f : X \rightarrow X \mid f \text{ a dimap } d - \text{homotopic to } id_X \text{ with } f(\sigma_{xy}) = \sigma'_{x'y'}\}$ .

It is also possible to view these endomorphism trace categories as 2-categories with points as objects, traces as 1-morphisms and  $d$ -self maps as 2-morphisms; it is easy to check that the interchange law holds.

#### 4. WEAKLY INVERTIBLE SYSTEMS AND COMPONENT CATEGORIES

**4.1. Motivation. An example.** The article [7] describes a method of "compressing" information in a small category by quotienting out a subcategory of so-called weakly invertible morphisms satisfying certain properties. This method has since been refined in [11] and related to a quotient approach based on general congruences on small categories that was described earlier in [2].

**Definition 4.1.** [11] A morphism  $\sigma \in \mathcal{C}(x, y)$  in a small category  $\mathcal{C}$  is called *Yoneda invertible* if, for every object  $z$  of  $\mathcal{C}$  with  $\mathcal{C}(y, z) \neq \emptyset$ , resp.  $\mathcal{C}(z, x) \neq \emptyset$  the maps  $\mathcal{C}(y, z) \xrightarrow{\sigma \circ} \mathcal{C}(x, z)$  and  $\mathcal{C}(z, x) \xrightarrow{\circ \sigma} \mathcal{C}(z, y)$  are bijections.

Yoneda invertible morphisms are at the base of the quotienting process described and applied to the fundamental category of a  $d$ -space in [7, 11]. While these seem to be very adequate for categories where all isomorphisms are identities (as in  $d$ -spaces arising from a partial order), the presence of loops causes serious problems.

**Example 4.2.** The simplest example is that of the oriented circle  $\vec{S}^1$  with  $\vec{P}(\vec{S}^1)$  consisting of the counterclockwise paths (arising from non-decreasing paths on the reals under the exponential map). In this case, for every pair of angles  $\alpha, \beta \in S^1$ , the *trace* space (after dividing out reparametrizations) is the discrete space represented by rotations by angles  $\beta - \alpha + 2k\pi, k \in \mathbf{Z}, k \geq 0$ . The trace category and the fundamental category (cf. Section 3.1) agree:  $\vec{\pi}_1(\vec{S}^1)(\alpha, \beta) = \vec{T}(\vec{S}^1)(\alpha, \beta) = \{\beta - \alpha + 2k\pi, k \in \mathbf{Z}, k \geq 0\}$ . Concatenation with a rotation by an angle  $\rho, 0 < \rho < 2\pi$ , corresponds to addition of angles and yields thus maps

$$\begin{array}{ccc} \vec{T}(\vec{S}^1)(\alpha, \beta) & \xrightarrow{+\rho} & \vec{T}(\vec{S}^1)(\alpha, \beta + \rho \bmod 2\pi) \\ \beta - \alpha + 2k\pi & \mapsto & \beta - \alpha + \rho + 2k\pi \end{array} \quad \begin{array}{ccc} \vec{T}(\vec{S}^1)(\alpha, \beta) & \xrightarrow{\rho+} & \vec{T}(\vec{S}^1)(\alpha - \rho \bmod 2\pi, \beta) \\ \beta - \alpha + 2k\pi & \mapsto & \beta - \alpha + \rho + 2k\pi. \end{array}$$

Remark that the addition of  $\rho$  is  $\bmod 2\pi$  on the objects and "on the nose" on the morphisms. In particular, for  $\beta - \alpha + \rho > 2\pi$ , neither of the maps  $+\rho$  or  $\rho+$  is surjective. Given any  $\beta \in S^1$  and  $\rho \neq 0$ , the morphism  $+\rho$  is certainly *not* a bijection for  $\alpha = \beta + \frac{\rho}{2}$  and the morphism  $\rho+$  is *not* a bijection for  $\beta = \alpha - \frac{\rho}{2}$ . In conclusion, the only Yoneda

invertible morphisms in the fundamental category of  $\vec{S}^1$  are the identities; nothing is gained by going over to a quotient category.

As a cure, we will consider the homotopy preorder category  $\vec{D}_\pi(\vec{S}^1)$ , cf. Section 3.2 of the trace=fundamental category. In this category, the morphism (using analogous notation)  $\vec{D}_\pi(\vec{S}^1)(\alpha, \beta) \xrightarrow{\rho+, +\sigma} \vec{D}_\pi(\vec{S}^1)(\alpha - \rho, \beta + \sigma)$  will be considered as weakly invertible if and only if  $\beta - \alpha + \rho + \sigma < 2\pi$ . Remark that these weakly invertible morphisms form a closed wide subcategory in the homotopy preorder category of  $\vec{S}^1$ . To be continued in Example 4.6.

**4.2. Weakly invertible morphisms with respect to a functor.** We now present a very general method of pointing out subcategories of weakly invertible morphisms: Consider a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between two small categories. A morphism  $\sigma \in \mathcal{C}(x, y)$  will be called *F-invertible* if and only if  $T(\sigma) \in \mathcal{D}(Fx, Fy)$  is an *isomorphism* in  $\mathcal{D}$ . Let  $\mathcal{C}_F(x, y) \subseteq \mathcal{C}(x, y)$  denote the set of all *F-invertible* morphisms from  $x$  to  $y$ . The collection of all  $\mathcal{C}_F(x, y)$  form a wide subcategory  $\mathcal{C}_F$  of  $\mathcal{C}$  since the composition of two *F-invertible* morphisms obviously is *F-invertible* again; remark that  $\mathcal{C}_F(x, y)$  contains the  $\mathcal{C}$ -isomorphisms.

For example, consider the functor  $\vec{T}^X : \vec{D}_\pi(X) \rightarrow Ho - Top$  or the functors  $\vec{H}_{n+1}(X) : \vec{D}_\pi(X) \rightarrow Ab$  from Sect. 3.2. A morphism  $(\sigma_x, \sigma_y) \in \vec{D}_\pi(X)((x, y), (x', y'))$  is  $\vec{T}^X$ -invertible if and only if  $\vec{T}(X)(\sigma_x, \sigma_y) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(x', y')$  is a homotopy equivalence; it is  $\vec{H}_{n+1}$ -invertible if  $(\sigma_x, \sigma_y)_* : H_n(\vec{T}(X)(x, y)) \rightarrow H_n(\vec{T}(X)(x', y'))$  is an isomorphism.

In Example 4.2, the weakly invertible morphisms on  $\vec{S}^1$  correspond exactly to the  $T^{\vec{S}^1}$ -invertible morphisms, which again are the same as the  $\vec{H}_1(\vec{S}^1)$ -invertible morphisms.

**4.3. Component categories.** Having defined the wide category  $\mathcal{C}_F$  of  $\mathcal{C}$ , we can proceed along the lines of [7] or of [11] to arrive at a quotient category identifying objects and morphisms that are linked to each other by  $\mathcal{C}_F$ -morphisms. In order to get a consistent construction, it is usually necessary to restrict the morphisms furthermore to a wide subcategory  $\Sigma \subseteq \mathcal{C}_F \subseteq \mathcal{C}$ :

**Definition 4.3.** [3, 7, 11] Let  $\Sigma \subseteq \mathcal{C}$  denote a wide subcategory of a small category  $\mathcal{C}$ . The pair  $(\mathcal{C}, \Sigma)$

**LEP/REP:** has the *left/right extension property* LEP/REP if and only if the diagrams

$$(4.1) \quad \begin{array}{ccc} x' & \xrightarrow{\tau'} & y' \\ \sigma \uparrow & & \uparrow \sigma' \\ x & \xrightarrow{\tau} & y \end{array} \quad \begin{array}{ccc} x & \xrightarrow{\tau} & y \\ \sigma' \uparrow & & \uparrow \sigma \\ x' & \xrightarrow{\tau'} & y' \end{array}$$

can be filled in with  $\tau' \in \mathcal{C}, \sigma' \in \Sigma$  given any  $\tau \in \mathcal{C}, \sigma \in \Sigma$ .

**pure:** is *pure* if and only if  $\sigma \circ \tau \in \Sigma \Leftrightarrow \sigma, \tau \in \Sigma$ ; (and left pure, resp. right pure, if one can only conclude  $\sigma \in \Sigma$ , resp.  $\tau \in \Sigma$ ).

**SLEP/SREP:** has the *strong left/right extension property* SLEP/SREP if the diagrams (4.1) can be filled in to yield pushout, resp. pullback squares in  $\mathcal{C}$ .

**Proposition 4.4.** [11] *Let  $\mathcal{B}$  denote a wide subcategory of  $\mathcal{C}$  and suppose that the pair  $(\mathcal{C}, \text{Iso}(\mathcal{C}))$  is pure.*

- (1) *If  $(\mathcal{C}, \mathcal{B})$  satisfies SLEP and SREP, then  $(\mathcal{C}, \mathcal{B})$  is pure.*
- (2) *The family of all wide subcategories  $\text{Iso}(\mathcal{C}) \subseteq \mathcal{D} \subseteq \mathcal{B} \subseteq \mathcal{C}$  such that  $(\mathcal{C}, \mathcal{D})$  satisfies SLEP and SREP is a complete lattice; in particular, there is a wide subcategory  $\Sigma_{\mathcal{B}} \subseteq \mathcal{B}$  such that  $(\mathcal{C}, \Sigma_{\mathcal{B}})$  satisfies SLEP and SREP and such that  $\mathcal{D} \subseteq \Sigma_{\mathcal{B}}$  for all  $\mathcal{D}$  above.*

There are now two possible points of departure from which to obtain a component category from a wide subcategory  $\text{Iso}(\mathcal{C}) \subseteq \Sigma \subseteq \mathcal{C}_F \subseteq \mathcal{C}$ :

- (1)  $\Sigma$  is a subcategory such that  $(\mathcal{C}, \Sigma)$  satisfies LEP/REP and purity; no maximality ensured.
- (2)  $\Sigma = \Sigma_{\mathcal{C}_F}$ , the maximal subcategory satisfying SLEP and SREP.

For the convenience of the reader, we include a brief description of the construction of the component category of  $\mathcal{C}$  with respect to  $\Sigma$  from [7] starting with the category of fractions  $\mathcal{C}[\Sigma^{-1}]$  [3, 7]. The exposition can be simplified since the extension properties from Def. 4.3 imply that the subcategory  $\Sigma$  admits a left, resp. right calculus of fractions [3, 7] on  $mc\mathcal{C}$ . A morphism in this category (with the same objects as those of  $\mathcal{C}$ ) is a ("zig-zag")-morphisms of the form  $\sigma^{-1} \circ \tau$ , resp.  $\tau \circ \sigma^{-1}$ ,  $\sigma \in \Sigma, \tau \in \mathcal{C}$ ; the extension properties make sure that the composite of two morphisms can be written in this "standard form" again. Morphisms of the form  $\sigma_1^{-1} \circ \sigma_2$ , resp.  $\sigma_1 \circ \sigma_2^{-1}$ ,  $\sigma_i \in \Sigma$  are the  $\Sigma$ -zig-zag morphisms [7]. The functor  $F$  maps  $\Sigma \subseteq \mathcal{C}_F$  into  $\text{Iso}(\mathcal{D})$  and can therefore be extended to a functor  $F[\Sigma^{-1}] : \mathcal{C}[\Sigma^{-1}] \rightarrow \mathcal{D}, \sigma^{-1} \circ \tau \mapsto (F(\sigma))^{-1} \circ F(\tau)$ ; likewise for  $F[\Sigma^{-1}]$ .

Two objects  $x, y$  of  $\mathcal{C}$  are called  $\Sigma$ -equivalent ( $x \simeq_{\Sigma} y$ ) if there exists a  $\Sigma$ -zig-zag-morphism between them. The equivalence classes are called the  $\Sigma$ -components of  $\mathcal{C}$ ; they are the path components *with respect to the  $\Sigma$ -zig-zag morphisms*. Moreover, we generate an equivalence relation on the morphisms of  $\mathcal{C}[\Sigma^{-1}]$  by requiring that  $\tau \simeq \tau \circ \sigma, \tau \simeq \sigma \circ \tau$  whenever  $\sigma \in \Sigma$  and the composition is defined.

The *component category*  $\pi_0(\mathcal{C}; \Sigma)$  of the preorder category has the  $\Sigma$ -components as objects; the morphisms from  $[x]$  to  $[y]$  are the equivalence classes of morphisms in  $\bigcup_{x' \simeq x, y' \simeq y} \mathcal{C}[\Sigma^{-1}]$ . Two morphisms in  $\pi_0(\mathcal{C}; \Sigma)$  represented by  $\tau_i \in \mathcal{C}(x_i, y_i), 1 \leq i \leq 2$  with  $y_1 \simeq_{\Sigma} x_2$  can be composed by inserting any  $\Sigma$ -zig-zag-morphism connecting  $y_1$  and  $x_2$ , cf. [7] for details.

Taking equivalence classes results in a functor  $q_{\Sigma} : \mathcal{C} \rightarrow \pi_0(\mathcal{C}; \Sigma)$ . By construction, the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  factors over  $q_{\Sigma}$  and the component category to yield a functor  $F : \pi_0(\mathcal{C}; \Sigma) \rightarrow \mathcal{D}$ .

The extension and pureness properties have the following consequences:

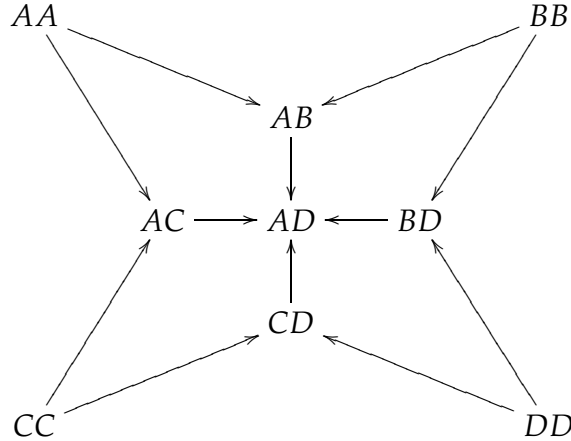
**Proposition 4.5.** (1) ([7], Proposition 5):  $(\Sigma, \Sigma)$  satisfies LEP/REP.

- (2) ([7], Proposition 3): Given a component  $C \subseteq \text{ob}(\mathcal{C})$  and elements  $x, y \in C$ . Every morphism  $\tau' \in \mathcal{C}(x', y')$  with  $x' \in C$  (resp.  $y' \in C$ ) is  $\Sigma$ -equivalent to a morphism  $\tau \in \mathcal{C}(x, -)$  (resp.  $\tau \in \mathcal{C}(-, y)$ ).
- (3) Every isomorphism in  $\pi_0(\mathcal{C}; \Sigma)$  is an endomorphism.
- (4) If  $\tau_1 \circ \tau_2 \in \pi_0(\mathcal{C}; \Sigma)(C, C)$  is an isomorphism, then the  $\tau_i, 1 \leq i \leq 2$ , are isomorphisms.

The last property is particularly important: it makes it impossible to leave and then reenter a component.

**Example 4.6.** (1) Let  $X$  denote the subspace of  $\bar{I}^2$  obtained by removing an (open) isothetic subsquare  $\bar{J}^2$ , cf. Figure 2. This space is divided into four components  $A$  (below the hole),  $B, C, D$  (above the hole). It is easy to see that the trace spaces  $\vec{T}(X)(x, y), x \preceq y$ , are homotopy equivalent to a discrete space consisting of *two* elements if and only if  $x \in A$  and  $y \in D$  and of *one* element else. Concatenation with a  $d$ -path is a homotopy equivalence unless the  $d$ -path crosses one of the borders of  $A$ , resp.  $D$  (the stipled lines in Figure 2.)

It is not difficult to see that the subcategory  $\Sigma(\vec{X}) \subseteq \vec{D}_\pi(X)$  consisting of pairs  $(\sigma_x, \sigma_y)$  with source and target in the same of the blocks  $A, B, C, D$  maps into homotopy equivalences under  $T^X$  and it satisfies SLEP/SREP. As a result, the component category  $\pi_0(\vec{D}_\pi(X); \vec{\Sigma}(X))$  can be depicted as follows (e.g.,  $AD$  is represented by a trace from a point in  $A$  to a point in  $D$ ):



The quadrilaterals from  $BB$  and from  $CC$  to  $AD$  commute, the other two quadrilaterals do not: there are two essentially different ways to extend a  $d$ -path starting and ending in  $A$  to a  $d$ -path starting in  $A$  and ending in  $D$ .

- (2) Let  $\vec{S}^1$  denote the oriented circle from Example 4.2. Let  $\Sigma(\vec{S}^1) \subseteq \vec{D}_\pi(\vec{S}^1)$  denote the wide subcategory consisting of all morphisms

$$\vec{D}_\pi \vec{S}^1(\alpha, \beta) \xrightarrow{\rho+, +\sigma} \vec{D}_\pi \vec{S}^1(\alpha - \rho, \beta + \sigma) \text{ with } \beta - \alpha + \rho + \sigma < 2\pi;$$

these are exactly the morphisms that are mapped to homotopy equivalences under  $\vec{T}^{\vec{S}^1}$ . Also in this case, it is easy to see that the pair of categories satisfies SLEP/SREP – by extending to the maximum of the targets/minimum of the sources.

The following diagram yields a zig-zag of  $\vec{\Sigma}(\vec{S}^1)$ -paths between arbitrary pairs of angles  $(\alpha, \beta), (\gamma, \delta)$ :

$$\begin{array}{ccccc}
 (\alpha, \beta) & & (\beta, \delta) & & (\gamma, \delta) \\
 \nwarrow (+\alpha - \beta, c_\beta) & \nearrow (c_\beta, +\delta - \beta) & \nwarrow (+\beta - \delta, c_\delta) & \nearrow (+\gamma - \delta, c_\delta) & \\
 (\beta, \beta) & & (\delta, \delta) & & 
 \end{array}$$

As a result, the component category  $\pi_0(\vec{D}_\pi(\vec{S}^1); \vec{\Sigma}(\vec{S}^1))$  has a single object and a morphism set in bijective correspondence with the non-negative integers generated by a single loop  $+2\pi$ .

**4.4. Component categories with respect to homotopy equivalences, homology, homotopy.** We will now apply the general construction of a component category from Section 4.3 to the case where  $\mathcal{C}$  is the homotopy preorder category or the homotopy factorization category of a  $d$ -space  $X$  and  $F$  is one of the functors  $T^X$ , resp. the homology and homotopy functors considered in Section 3). In both the preorder and the factorization category, only the identities are isomorphisms. Moreover, the factorization category is loopfree (This property is essential for the later parts of [11].)

In these cases,  $C_F$  will be the subcategory consisting of concatenation morphisms that induce homotopy equivalences, resp. induce isomorphisms on certain homology or homotopy groups.

We will indicate later in Sect. 5 a topologically natural way to ensure LEP/REP. The delicate point is then to choose a suitable subcategory  $\Sigma \subseteq C_F$  satisfying pureness, in addition. It is not so clear to me how natural it is to require SLEP/SREP (with pureness as a consequence), and case studies (in particular in spaces/categories with non-trivial loops) should be performed.

Having chosen a suitable such subcategory  $\Sigma_F \subseteq C_F$ , we can construct the component categories  $\pi_0(\vec{D}_\pi(X), \Sigma_F)$ . This leads to a consistent way of identifying pairs  $(x, y), (x', y')$  in the preorder category with  $\vec{T}(X)(x, y), \vec{T}(X)(x', y')$  being homotopy equivalent, homology equivalent in certain dimensions etc. We obtain factorizations of the original functors

$$\begin{aligned}
 \vec{T}_\pi^X : \vec{D}_\pi(X) &\xrightarrow{q_{\Sigma_T}} \pi_0(\vec{D}_\pi(X), \Sigma_T) \xrightarrow{\pi_0(\vec{T}_\pi^X)} Ho - Top, \\
 \vec{H}_{n+1}(X) : \vec{D}_\pi(X) &\xrightarrow{q_{\Sigma_H}} \pi_0(\vec{D}_\pi(X), \Sigma_H) \xrightarrow{\pi_0(\vec{H}_{n+1}(X))} Ab, \\
 \vec{\pi}_{n+1}(X) : F\vec{T}_\pi(X) &\xrightarrow{q_{\Sigma_\pi}} \pi_0(F\vec{T}_\pi(X), \Sigma_\pi) \xrightarrow{\pi_0(\vec{\pi}_{n+1}(X))} Sets/Grps/Ab.
 \end{aligned}$$

*Remark 4.7.* A similar and somewhat simpler situation has been investigated under the heading “persistent homology” by G. Carlsson and collaborators, in particular for applications in the analysis of statistical point cloud data, c.f. e.g. [4, 21]. The indexing category corresponds to the preorder category of ordinals  $1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$ . A (homology) functor from this category into  $k$ -modules,  $k$  a field, is a  $k[t]$ -module and splits as such into irreducible modules of the form  $\Sigma^i k[t]$  or  $\Sigma^i k[t]/(t^j)$  of classes that are “born at step”  $i$  and possibly annihilated after  $j$  steps. These are denoted by “barcodes” with bars from  $i$  to  $\infty$  or to  $i + j$ . In our case, we can have homology classes that are born at  $(x, y)$  (in the cokernel of maps into  $\vec{H}_*(X)(x, y)$ ) that survive to  $(x', y')$  and are annihilated at  $(x'', y'')$  (in the kernel of the map into  $\vec{H}_*(X)(x'', y'')$ ). These births and deaths are obstructions to invertibility with respect to the functors  $\vec{H}_*$ .

## 5. AUTOMORPHIC HOMOTOPY FLOWS

This section prepares a new definition for directed homotopy equivalences and an investigation of their properties. Which requirements should a  $d$ -map  $f : X \rightarrow Y$  satisfy in order to qualify as a directed homotopy equivalence? Obviously, there should be a reverse  $d$ -map  $g : Y \rightarrow X$  such that both  $g \circ f$  and  $f \circ g$  are  $d$ -homotopic to the resp. identity maps.<sup>1</sup> But this is not enough: The ( $d$ -path) structures on  $X$  and  $Y$  ought to be homotopically related, i.e., the maps  $\vec{T}(f) : \vec{T}(X)(x, y) \rightarrow \vec{T}(Y; fx, fy)$  should be ordinary homotopy equivalences – for all  $x, y$  with  $\vec{T}(X)(x, y) \neq \emptyset$  – and that in a natural way.

*Remark 5.1.* (1) For the lpo-spaces of [8] one might instead ask, that the “intervals”  $[x, y]$ , resp.  $[fx, fy]$  containing all points between  $x$  and  $y$ , resp. between  $fx$  and  $fy$  are homotopy equivalent.  
 (2) Compare with the future, resp. past homotopy equivalence from [16]; note also the coherence requirements there.

This is in general not the case: A first indication for this is that a self- $d$ -map  $h : X \rightarrow X$  that is  $d$ -or di-homotopic to the identity map  $id_X : X \rightarrow X$  does *not* always yield homotopy equivalences  $\vec{T}(X)(x, y) \rightarrow \vec{T}(X)(hx, hy)$ , cf. Example 5.10 below.

To get an idea of what a dihomotopy equivalence should satisfy – and also for a suggestion on subcategories related to components, one needs to understand  $d$ -homotopies of the identity  $id_X$  of a  $d$ -space  $X$ .

**Definition 5.2.** (1) A  $d$ -map  $H : X \times \vec{I}$  to  $X$  is called a future homotopy flow, if  $H_0 = id_X$  and a past homotopy flow, if  $H_1 = id_X$ .  
 (2) The sets consisting of all future homotopy flows, resp. of all past homotopy flows will be denoted by  $\vec{P}_+C_0(X, X)$ , resp. by  $\vec{P}_-C_0(X, X)$ .

---

<sup>1</sup>One may moreover ask for future/past homotopy equivalence as in [16]; note also the coherence requirements in that paper.



Homotopy flows (and the later refinements) generalise the concept of a *flow* on a differentiable manifold. We do not require that the maps  $H(-, t) : X \rightarrow X$  are homeomorphisms. The orbits of a flow have the following counterpart: For every  $x \in X$ , the map  $H_x : \vec{I} \rightarrow X$ ,  $t \mapsto H(x, t)$  is a  $d$ -path (with  $H_x(0) = x$ , resp.  $H_x(1) = x$ ). Evaluation at  $x \in X$  defines maps

$$(5.1) \quad ev_+^x : \vec{P}_+C_0(X, X) \rightarrow \vec{T}(X)(x, -), \text{ resp. } ev_-^x : \vec{P}_-C_0(X, X) \rightarrow \vec{T}(X)(-, x).$$

A maximal element  $x_+ \in X$  – with the constant path as the only  $d$ -path with source  $x_+$  – will be fixed under a future homotopy flow, likewise a minimal element under a past homotopy flow.

Remark that future homotopy flows can be pieced together in various natural ways:

$$(5.2) \quad (H_1 * H_2)(x, t) = \begin{cases} H_1(x, 2t), & t \leq \frac{1}{2}, \\ H_2(H_1(x, 1), 2t - 1), & t \geq \frac{1}{2}, \end{cases} \text{ resp. } (H_1 \square H_2)(x, t) = H_2(H_1(x, t), t).$$

Similarly for past homotopy flows. In particular, if  $f, g : X \rightarrow X$  are future, resp. past  $d$ -homotopic to  $id_X$ , then their compositions  $f \circ g : X \rightarrow X$  and  $g \circ f : X \rightarrow X$  are so, as well; they form thus sub-monoids of the monoid  $\vec{C}(X, X)$  of all self- $d$ -maps of  $X$ . Likewise for *traces* of past/future homotopy flows (quotienting out “global” reparametrizations).

Homotopy flows induce several interesting maps on trace spaces: Let  $H_+, H_- : X \times \vec{I} \rightarrow X$  denote  $d$ -homotopies  $id_X \rightarrow f$ , resp.  $g \rightarrow id_X$ . These  $d$ -homotopies define, for every  $x \in X$  the  $d$ -paths  $H_{\pm x}$  from  $x$  to  $fx$ , resp. from  $gx$  to  $x$ . For reasons to be explained in Rem. 5.7 below, we need moreover to consider restrictions of the homotopies and their effect on trace spaces: for every  $s \in I$ , there is a (restricted)  $d$ -homotopy  $H_{\pm}^s : X \times \vec{I} \rightarrow X$ ,  $H_{\pm}^s(x, t) = H_{\pm}(x, st)$  with associated  $d$ -paths  $H_{\pm x}^s$  from  $x$  to  $H_+(x, s)$ , resp. from  $H_-(x, s)$  to  $x$ . As explained in Sect. 3, the maps  $H_{\pm}^s, H_{+y}^s, H_{-x}^s$  induce maps on trace spaces, that are linked to each other by the following homotopy commutative diagrams:

$$(5.3) \quad \begin{array}{ccc} \vec{T}(X)(x, y) & \xrightarrow{\vec{T}^X(c_x, H_{+y}^s)} & \vec{T}(X)(x, H_+(y, s)) \\ \downarrow \vec{T}(H_+^s) & & \uparrow \vec{T}^X(H_{+x}^s, c_{H_+(y, s)}) \\ \vec{T}(X)(H_+(x, s), H_+(y, s)) & & \end{array} \quad \begin{array}{ccc} \vec{T}(X)(x, y) & \xrightarrow{\vec{T}^X(H_{-x}^s, c_y)} & \vec{T}(X)(H_-(x, s), y) \\ \downarrow \vec{T}(H_-^s) & & \uparrow \vec{T}^X(c_{H_-(x, s)}, H_{-y}^s) \\ \vec{T}(X)(H_-(x, s), H_-(y, s)) & & \end{array}$$

(The maps  $\vec{T}^X(-, -)$  are defined in Sect. 3.2 and arise from concatenation). Commutativity in the diagram is a consequence of a more general

**Lemma 5.3.** *A  $d$ -homotopy  $H : X \times \vec{I} \rightarrow Y$  induces, for every object  $(x, y)$  in  $\vec{D}(X)$ , a  $d$ -homotopy  $\vec{H}(x, y) : \vec{T}(X)(x, y) \times I \rightarrow \vec{T}(Y)(H(x, 0), H(y, 1))$  between  $\vec{H}_0(x, y)(\sigma) = H(\sigma, 0) * H(y, t)$  and  $\vec{H}_1(x, y)(\sigma) = H(x, t) * H(\sigma, 1)$ .*

A similar construction has been used in [6].

*Proof.* Consider the  $d$ -map  $K : \vec{I}^2 \rightarrow \vec{I}^2$  given by

$$K(s, t) = \begin{cases} (0, 3st) & \text{for } t \leq \frac{1}{3} \\ (3t - 1, s) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3} \\ (1, 3(t + s - st)) - 2 & \text{for } \frac{2}{3} \leq t, \end{cases}$$

cf. Figure 1. Remark that  $K(s, 0) = (0, 0)$ ,  $K(s, 1) = (1, 1)$ ,  $s \in I$ , and that  $K(0, t)$ ,  $K(1, t)$  are parametrizations of the two  $d$ -paths connecting  $(0, 0)$  with  $(1, 1)$  along the boundary  $\partial \vec{I}^2$  of  $\vec{I}^2$ .

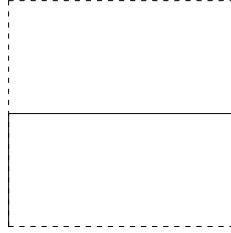


FIGURE 1. The path  $K(\frac{1}{2}, t)$

Composition  $H \circ K$  defines a  $d$ -homotopy (with fixed boundary) connecting a reparametrization of  $\vec{H}_0(x, y)(\sigma)$  to a reparametrization of  $\vec{H}_1(x, y)$ .  $\square$

**Definition 5.4.** A future/past homotopy flow  $H : X \times \vec{I} \rightarrow X$  is called

- (1) *automorphic* if, for all  $x, y \in X$  with  $\vec{T}(X)(x, y) \neq \emptyset$  and all  $s \in I$ , the maps  $\vec{T}(H_+^s)$ , resp.  $\vec{T}(H_-^s)$  (vertical in (5.3)) are *homotopy equivalences*;
- (2) The sets consisting of all automorphic future/past homotopy flows will be denoted by  $\vec{P}_\pm \text{Aut}(X)$ .
- (3) A self- $d$ -map  $f : X \rightarrow X$  is called a future/past-automorphism if there exists an automorphic future/past homotopy flow between  $f$  and the identity on  $X$ . The set of all future/past-automorphisms on  $X$  will be denoted  $\text{Aut}_*(X) \subseteq \vec{C}(X, X)$ ,  $* = +, -$ .

*Remark 5.5.* (1) In particular, the maps

$$\vec{T}(f) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(fx, fy) \text{ resp. } \vec{T}(g) : \vec{T}(X)(gx, gy) \rightarrow \vec{T}(X)(x, y)$$

are homotopy equivalences.

- (2) Using the concatenation  $*$  of homotopy flows from (5.2), it is obvious that *automorphisms* form *submonoids* of  $\vec{C}(X, X)$ .
- (3) The definitions above come close to that of a flow on a manifold. But remark again, that the maps  $H(-, t) : X \rightarrow X$  are not required to be homeomorphic; in particular, they will in general not be invertible.

**Lemma 5.6.** *Let  $H$  denote a future/past homotopy flow on  $X$ .*

- (1) *If all (skew) concatenation maps in (5.3) are homotopy equivalences, then  $H$  is automorphic.*
- (2) *Let  $H$  be automorphic. If one of the (skew) concatenation maps in (5.3) is a homotopy equivalence, then the other is as well.*

*Proof.* Immediate from (5.3). □

**Remark 5.7.** It is in general not enough to ask that the maps induced by the entire  $d$ -homotopy  $H$  in (5.3) are homotopy equivalences. In general, one cannot conclude that the maps induced by  $H^s$  are homotopy equivalences, as well. We need that finer requirement crucially in the discussion of components in Sect. 7.

**Remark 5.8.** In the undirected case, it is unnecessary to ask homotopies to be automorphic : if  $x, y, x', y' \in X$  are in the same path-component of a topological space, then the sets of paths  $P(X)(x, y)$  and  $P(X)(x', y')$  are always homotopy equivalent to each other.

One may also study the effects of the maps induced by a homotopy flow under the homology, resp. homotopy functors from Sect. 3 ( $n > 0$  and  $\vec{T}(X)(x, y) \neq \emptyset$ ):

(5.4)

$$\begin{array}{ccc}
 \vec{H}_{n+1}(X)(x, y) & \xrightarrow{\vec{T}^X(c_x, H_{+y}^s)_*} & \vec{H}_{n+1}(X)(x, H_+(y, s)) \\
 \downarrow \vec{T}(H_+^s)_* & & \uparrow \vec{T}^X(H_{+x}^s, c_y)_* \\
 \vec{H}_{n+1}(X)(H_+(x, s), H_+(y, s)) & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \vec{H}_{n+1}(X)(x, y) & \xrightarrow{\vec{T}^X(H_{-x}^s, c_y)_*} & \vec{H}_{n+1}(X)(H_-(x, s), y) \\
 \downarrow \vec{T}(H_-^s)_* & & \uparrow \vec{T}^X(c_x, H_{-y}^s)_* \\
 \vec{H}_{n+1}(X)(H_-(x, s), H_-(y, s)) & & 
 \end{array}$$

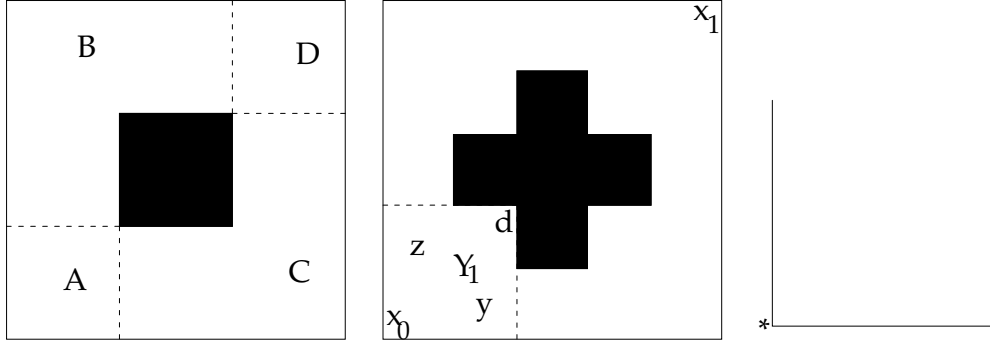
and

(5.5)

$$\begin{array}{ccc}
 \vec{\pi}_{n+1}(X)(\sigma_{xy}) & \xrightarrow{\vec{T}^X(c_x, H_{+y}^s)_\#} & \vec{\pi}_{n+1}(X)(\sigma_{xy} * H_{+y}^s) \\
 \downarrow \vec{T}(H_+^s)_\# & & \uparrow \vec{T}^X(H_{+x}^s, c_y)_\# \\
 \vec{\pi}_{n+1}(X)(H_+^s \sigma_{xy}) & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \vec{\pi}_{n+1}(X)(\sigma_{xy}) & \xrightarrow{\vec{T}^X(H_{-x}^s, c_y)_\#} & \vec{\pi}_{n+1}(X)(H_{-x}^s * \sigma_{xy}) \\
 \downarrow \vec{T}(H_-^s)_\# & & \uparrow \vec{T}^X(c_x, H_{-y}^s)_\# \\
 \vec{\pi}_{n+1}(X)(H_-^s \sigma_{xy}) & & 
 \end{array}$$

An automorphic homotopy flow induces bijections/group isomorphisms  $\vec{T}(H^s)_*$  on  $\vec{H}_*(X)$ , resp.  $\vec{T}(H^s)_\#$  on  $\vec{\pi}_*(X)$ . Sometimes, a weaker requirement can do (and will be used in Sect.7):

**Definition 5.9.** (1) A homotopy flow is said to be *automorphic up to homology*, if it induces isomorphisms on all relevant homology groups in (5.4).

FIGURE 2. The po-spaces  $X$  and  $S$  from Ex. 5.10 and  $L$  from Ex. 6.3

- (2) A homotopy flow is said to be automorphic up to homotopy/homology *in or up to a fixed dimension  $k$*  if it induces isomorphisms on all homotopy/homology sets/groups in (5.5)/(5.4) *in or up to dimension  $k$* .
- (3) Spaces of such homotopy flows are denoted by an additional decoration, we write e.g.  $\vec{P}_{\pm} \text{Aut}_{H_{\leq k}}(X)$ .

Again, maps that are future/past  $d$ -homotopic to the identity by a  $d$ -homotopy satisfying one of the requirements above, form a submonoid of  $\vec{C}(X, X)$ .

**Example 5.10.** (1) Let  $X$  denote the  $d$ -space (square with a hole) from Ex. 4.6. A future homotopy flow will always preserve  $A$  and  $D$ , but it may move elements of both  $B$ , resp.  $C$  into  $D$ . An *automorphic* future homotopy flow does *not* allow this: Let  $f : X \rightarrow X$  denote a  $d$ -map,  $x, fx \in A, y \in B, fy \in D$ . Then  $\vec{T}(X)(x, y)$  is contractable whereas  $\vec{T}(X)(fx, fy)$  consists of two path-components. Hence, there cannot exist an automorphic future homotopy flow  $H : id_X \rightarrow f$ .

(2) Let  $S$  denote the “Swiss flag” po-space[8], cf. the drawing in the middle of Fig. 2. By a combination of a future and a past homotopy flow, the identity on  $S$  is  $d$ -homotopic to a map that sends  $X$  to the 1-skeleton of the outer square; the area  $Y_1$  and in particular the “deadlock”  $d$  will be sent to the minimal element  $x_0$ . But the dihomotopies involved are *not* automorphic. It is important for applications in concurrency theory that the Swiss-flag space and its outer boundary should *not* be considered as equivalent: Deadlocks should not disappear under an equivalence!

## 6. DIHOMOTOPY EQUIVALENCES

### 6.1. Definitions.

**Definition 6.1.** (1) A  $d$ -map  $f : X \rightarrow Y$  is called a *future dihomotopy equivalence* if there exist  $d$ -maps  $f_+ : X \rightarrow Y, g_+ : Y \rightarrow X$  such that  $f, f_+$  are  $d$ -homotopic and *automorphic*  $d$ -homotopies  $H^X : id_X \rightarrow g_+ \circ f_+$  on  $X$  and  $H^Y : id_Y \rightarrow f_+ \circ g_+$  on  $Y$ .

- (2) The  $d$ -map  $f : X \rightarrow Y$  is called a *past dihomotopy equivalence* if there exist  $d$ -maps  $f_- : X \rightarrow Y, g_- : Y \rightarrow X$  such that  $f, f_-$  are  $d$ -homotopic and *automorphic*  $d$ -homotopies  $H^X : g_- \circ f_- \rightarrow id_X$  on  $X$  and  $H^Y : f_- \circ g_- \rightarrow id_Y$  on  $Y$ .
- (3) The  $d$ -map  $f$  is called a *dihomotopy equivalence* if it is *both* a future and a past dihomotopy equivalence.

*Remark 6.2.* (1) The requirements in Definition 6.1 above should be seen as a requirement to a  $d$ -homotopy *class* of dimaps from  $X$  to  $Y$ .

- (2) Example 5.10 shows that we ask for more than just the existence of an inverse up to  $d$ -homotopy.
- (3) Only the essential extra (automorphism) requirement allows to arrive at valuable conclusions regarding effects on the trace, fundamental, homotopy and homology categories; cf. Prop. 6.4 below.
- (4) A similar requirement is unnecessary in the classical undirected case; cf. Remark 5.8.
- (5) Our definition above is related to the definition of a *faithful* future, resp. past equivalence in the work of Grandis[16] between (general) categories. For a category like the fundamental category  $\mathcal{C} = \vec{\pi}_1(X)$  or the trace category  $\mathcal{C} = \vec{T}(X)$ , faithfulness amounts to asking that the map induced by right concatenation with  $H_y^X$  from  $\mathcal{C}(x, y)$  to  $\mathcal{C}(x, gfy)$  is *epi* and *mono* within the category  $\mathcal{C}$ . This requirement has nice consequences (cf. the Cancellation Lemma 2.2 in [16], which requires an extra coherence condition), but it is restricted to what can be seen from within the category  $\mathcal{C}$ . Our requirements are expressed with respect to a functor (e.g.,  $\vec{T}^X$ ) to a *different* category (e.g.  $Ho - Top$ ).
- (6) An argument similar to that used for *pf*-equivalences in [16], Section 3, shows that the “homotopy inverses”  $g_+, g_-$  of a dihomotopy equivalence are  $d$ -homotopic to each other:  $g_- \mapsto g_+ \circ f_+ \circ g_-$  (by  $H_+^X$ ),  $g_+ \circ f_+ \circ g_-$  is  $d$ -homotopic to  $g_+ \circ f_- \circ g_-$ , and  $g_+ \circ f_- \circ g_- \mapsto g_+$  (by  $H_-^Y$ ). Another  $d$ -homotopy is given by  $g_- \mapsto g_- \circ f_+ \circ g_+ \simeq g_- \circ f_- \circ g_+ \mapsto g_+$ . Grandis requires  $f = f_+ = f_-$  and, for coherence, that the two compositions above agree.

**Example 6.3.** Here is a simple example of a past dihomotopy equivalence that is *not* future: Let  $L$  denote a “branching” po-space (a subspace of Euclidean space with induced partial order and hence  $d$ -space structure) in the shape of the letter  $L$  with base point  $*$  as its lower left vertex, cf. drawing in Fig. 2 on the right.

Inclusion  $i : \{*\} \rightarrow L$  and the constant map  $c : L \rightarrow \{*\}$  satisfy  $c \circ i = id_*$ ; moreover, there is an (increasing) dihomotopy  $i \circ c \rightarrow id_L$ . But for *no* map  $i_+ : \{*\} \rightarrow L$  does there exist an (increasing) dihomotopy  $id_L \rightarrow i_+ \circ c$ . Another way to phrase this is: The space  $L$  is *past contractible*, but *not future contractible*. Compare [14]. It is crucial for applications in concurrency that dihomotopy equivalence distinguish between a branching and a non-branching space.

In general, a  $d$ -space with more than one local maximum cannot be future contractible; if it has more than one local minimum, it cannot be past contractible.

**Proposition 6.4.** *The natural transformation  $\vec{T}_\pi(f) : \vec{T}_\pi^X \rightarrow \vec{T}_\pi^Y$  induced by a (past or future) dihomotopy equivalence  $f : X \rightarrow Y$  between  $d$ -spaces  $X$  and  $Y$  is an equivalence, i.e., the induced maps  $\vec{T}(f)(x, y) : \vec{T}(X)(x, y) \rightarrow \vec{T}(Y)(fx, fy)$  are homotopy equivalences.*

$$\begin{array}{ccccccc} \vec{T}(X)(x, y) & \xrightarrow{\vec{T}(f)} & \vec{T}(Y)(fx, fy) & \xrightarrow{\vec{T}(g)} & \vec{T}(X)(gfx, gfy) & \xrightarrow{\vec{T}(f)} & \vec{T}(Y)(fgfx, fgfy), \\ & & \swarrow I & & \searrow J & & \\ & & & & & & \end{array}$$

**Example 6.5.** The “Swiss flag” space  $S$  cannot be dihomotopy equivalent to a graph  $G$ : With reference to the middle drawing in Fig. 2, assume  $f : S \rightarrow G$  is a dihomotopy equivalence. By Prop. 6.4, there are unique directed paths  $\rho_y : fx_0 \rightarrow fy$  (from  $fx_0$  to  $fy$ ),  $\rho_z : fx_0 \rightarrow fz$ ,  $\sigma_y : fy \rightarrow fd$ ,  $\sigma_z : fy \rightarrow fd$ ,  $\tau_y : fy \rightarrow fx_1$ ,  $\tau_z : fz \rightarrow fx_1$  and

There is only one directed connection from  $fx_0$  to  $fd$  in  $G$ . Hence, we can assume without restriction of generality, that  $\rho y$  is a "prefix" of  $\rho_z$ , i.e., there exists  $\rho_{yz} : fy \rightarrow fz$  such that  $\rho_z = \rho_y * \rho_{yz}$ . But then  $\rho_z * \tau_z = \rho_y * \rho_{yz} * \tau_z = \rho_y * \tau_y$ . This contradicts (6.1)!

$$(6.2) \quad \mathcal{H}^X : \vec{T}(Y)(fx, y) \xrightarrow{\vec{T}(g)} \vec{T}(X)(gfx, gy) \xrightarrow{\vec{T}^X(H_{+x}, c_{gy})} \vec{T}(X)(x, gy), \quad \alpha \mapsto H_{+x} * g\alpha,$$

For past equivalences, the respective maps are given by  $\alpha \mapsto g\alpha * H_{-x}$ , resp.  $\beta \mapsto f\beta * H_{-y}$ . These maps are homotopy equivalences if and only if the maps  $\vec{T}^X(H_{+x}, c_{gy})$  are homotopy equivalences.

**Proposition 6.6.** *The composition  $g \circ f : X \rightarrow Z$  of (future or past) dihomotopy equivalences  $X \xrightarrow{f} Y \xrightarrow{g} Z$  is again a (f/p) dihomotopy equivalence.*

*Proof.* Let  $Z \xrightarrow{g'} Y \xrightarrow{f'} X$  denote homotopy inverses to  $f$ , resp.  $g$ , and let  $H^X : id_X \rightarrow f' \circ f$  and  $H^Y : id_Y \rightarrow g' \circ g$  denote strictly automorphic  $d$ -homotopies. By a slight abuse of notation, let  $f' \circ H^Y \circ f : f' \circ f \rightarrow f' \circ g' \circ g \circ f$  denote the induced  $d$ -homotopy on  $X$ .

The composition  $\bar{H} : id_X \xrightarrow{H^X} f' \circ f \xrightarrow{f' \circ H^Y \circ f} f' \circ g' \circ g \circ f$  is a homotopy flow; we have to show that it is automorphic. The "levels"  $\bar{H}_s$  of  $\bar{H}$  are either of the form  $H_s^X$  – which induce homotopy equivalences by definition – or of the form  $f' \circ H_s^Y \circ f$ .

Let  $I : \vec{T}(X)(f'fx, f'fy) \rightarrow \vec{T}(X)(x, y)$  denote a homotopy inverse to  $\vec{T}(f') \circ \vec{T}(f)$ , let  $J : \vec{T}(Y)(ff'fx, ff'fy) \rightarrow \vec{T}(Y)(fx, fy)$  denote a homotopy inverse to  $\vec{T}(f) \circ \vec{T}(f')$ , and let  $K_s : \vec{T}(Y)(H_s^Y(fx), fH_s^Y(fy)) \rightarrow \vec{T}(Y)(fx, fy)$  denote a homotopy inverse to  $\vec{T}(H_s^Y)$ . Then  $\vec{T}(f') \circ \vec{T}(H_s^Y) \circ \vec{T}(f)$  has homotopy right and left inverses:

- $(I \circ \vec{T}(f') \circ K_s \circ J \circ \vec{T}(f)) \circ (\vec{T}(f') \circ \vec{T}(H_s^Y) \circ \vec{T}(f)) \simeq id$  on  $\vec{T}(X)(x, y)$ ;
- $(\vec{T}(f') \circ \vec{T}(H_s^Y) \circ \vec{T}(f)) \circ (\vec{T}(f') \circ J \circ K_s \circ \vec{T}(f) \circ I) \simeq id$  on  $\vec{T}(X)(x, y)$ ,

which have to agree by general nonsense.  $\square$

## 7. AUTOMORPHIC HOMOTOPY FLOWS AND COMPONENTS

**7.1. Motivation.** The localization construction from Section 4 has a drawback: It is in general difficult to get hold on a (preferably large) subcategory  $\Sigma$  of the original category that both maps into isomorphisms and satisfies the extension properties (and hopefully also pureness). The construction in [11] focussing on the strong extension properties is categorically very satisfactory, but it is not clear that the resulting category will be "large enough" to yield satisfactory compression.

The construction below is of a more "geometric nature" and uses the automorphic homotopy flows introduced in Section 5. On the positive side, the (non-strong) extension properties follow right away. Moreover, the set-up is very much related to the dihomotopy equivalences from Section 6, a fact that will be exploited in the final section 8. On the negative side, it seems not to be possible to prove that the resulting subcategory is pure and Prop. 4.5.(4) need not always be satisfied: the associated component category can therefore have isomorphisms that split into non-isomorphisms (which may leave component of the start point).

**7.2. The extended preorder category.** We extend the preorder category  $\vec{D}_\pi(X)$  to a category  $\vec{D}_\pi^{Aut_+}(X)$  with the same objects but with more morphisms. The morphisms in this new category are generated by those from the previous and additionally by morphisms  $f(x, y)$  from  $(x, y)$  to  $(fx, fy)$  for every  $f \in Aut(X_+)$  and every  $x \preceq y$  subject to the following relations (compare (5.3) for every automorphic future homotopy flow

$H_+$ , resp. every past homotopy flow  $H_-$ ):

$$(7.1) \quad \begin{array}{ccc} (fx, fy) & \xrightarrow{(c_{fx}, f\sigma)} & (fx, fz) \\ f(x,y) \uparrow & & \uparrow f(x,z) \\ (x, y) & \xrightarrow{(c_x, \sigma)} & (x, z) \end{array} \quad \begin{array}{ccc} (fx, fy) & \xrightarrow{(f\tau, c_{fy})} & (fu, fy) \\ f(x,y) \uparrow & & \uparrow f(u,y) \\ (x, y) & \xrightarrow{(\tau, cy)} & (u, y) \end{array}$$

for  $\sigma \in \vec{T}(X)(y, z)$ ,  $\tau \in \vec{T}(u, x)$ , and

$$(7.2) \quad \begin{array}{ccc} (x, y) & & (x, y) \\ \downarrow H_+(x,y) & \searrow (c_x, H_+y) & \searrow (H_-x, cy) \\ & (x, H_+(y)) & (H_-(x), y) \\ \uparrow (H_+x, c_{H_+(y)}) & & \uparrow (c_{H_-(x)}, H_-y) \\ (H_+(x), H_+(y)) & & (H_-(x), H_-(y)) \end{array}$$

for a an automorphic future homotopy flow  $H_+$ , resp. a past homotopy flow  $H_-$ .

By abuse of notation, we let  $Aut_+(X)$  denote the wide subcategory of  $\vec{D}_\pi^{Aut_+}(X)$  with morphisms stemming from automorphisms alone; this is in fact a subcategory, since  $Aut_+(X)$  is closed under concatenation. There is an obvious variant  $Aut_-(X)$  giving rise to an extended preorder category  $\vec{D}_\pi^{Aut_-}(X)$ . Considering future and past homotopy flows simultaneously leads to the category  $Aut(X)$  with morphisms  $(f_+, f_-)(x, y) : (x, y) \rightarrow (x', y')$  with  $f_+(x) = x'$ ,  $f_+(y) = y'$ ,  $f_-(x') = x$ ,  $f_-(y') = y$  such that  $\vec{T}(f_-f_+) \simeq id$  on  $\vec{T}(X)(x, y)$  and  $\vec{T}(f_+f_-) \simeq id$  on  $\vec{T}(X)(x', y')$ . The remainig part of this section are formulated for  $Aut(X)$ , but most of it applies also for the future, resp. past versions.

- Proposition 7.1.** (1) Every element of  $\vec{D}_\pi^{Aut_+}(X)$  can be written in the form  $c \circ f$  with  $c$  a morphism in the preorder category and  $f \in Aut(X)$ .  
(2) Within  $\vec{D}_\pi^{Aut}(X)$ , the subcategory  $Aut(X)$  satisfies LEP/REP with respect to  $\vec{D}_\pi(X)$  – explanation in the proof.  
(3) The functor  $\vec{T}^X : \vec{D}_\pi(X) \rightarrow Ho - Top$  extends to  $\vec{D}_\pi^{Aut}(X)$  and maps  $Aut(X)$  into isomorphisms.

*Proof.* (1) Successive application of (7.1) applied to a “mixed” morphism.

- (2) For  $f_+ \in Aut_+(X)$ ,  $f_- \in Aut_-(X)$ ,  $\tau \in \vec{T}(X)(x, y)$ , we can fill in extension diagrams as follows:

$$\begin{array}{ccc} (f_+x, f_+y) & \xrightarrow{(c_{f_+x}, f_+\tau)} & (f_+x, f_+y') \\ f_+(x,y) \uparrow & & \uparrow f_+(x,y') \\ (x, y) & \xrightarrow{(c_x, \tau)} & (x, y') \end{array} \quad \begin{array}{ccc} (x, y) & \xrightarrow{\tau} & (x, y') \\ f_-(x,y') \uparrow & & \uparrow f_-(x,y) \\ (f_-x, f_-y) & \xrightarrow{(c_{f_-x}, f_-\tau)} & (f_-x, f_-y') \end{array}$$



- (3)  $\vec{T}^X$  maps  $f(x, y) : (x, y) \rightarrow (fx, fy)$  into the homotopy equivalence  $\vec{T}(f)(x, y) : \vec{T}(X)(x, y) \rightarrow \vec{T}(X)(fx, fy)$ . The relations in (7.1, 7.2) are respected.  $\square$

*Remark 7.2.* The coherent automorphisms (cf. Definition 7.6) satisfy LEP/REP with respect to all of  $\vec{D}_\pi^{Aut}(X)$ .

**7.3. Category of fractions and components.** Now we construct a component category from the pair  $(\vec{D}_\pi^{Aut}(X), Aut(X))$  along the outline from Section 4.3. Since Proposition 7.1(2) only leads to solutions of extension properties with respect to the subcategory  $\vec{D}_\pi(X)$ , elements of the category of fractions  $\vec{D}_\pi^{Aut}(X)[Aut(X)^{-1}]$  will in general have a normal form of the type  $c \circ g$  with  $g = g_1 \circ h_1^{-1} \circ \dots \circ g_n \circ h_n^{-1} \in Aut(X)^{-1}$ . Still, the component category  $\pi_0(\vec{D}_\pi^{Aut}(X), Aut(X))$  is well-defined, coming with a factorization

$$\vec{T}_\pi(X) : \vec{D}_\pi^{Aut}(X) \xrightarrow{q_{Aut}} \pi_0(\vec{D}_\pi^{Aut}(X), Aut(X)) \xrightarrow{\pi_0(\vec{T}_\pi(X))} Ho - Top.$$

More serious is the following fact: Morphisms in the original preorder category  $\vec{D}_\pi(X)$  can contribute to the isomorphisms in the category of fractions: For example, assume that  $h = g \circ f \in Aut(X)$ ,  $H : id_X \rightarrow g$  with  $gfx = fx$  and  $\tau = H_{fy} \in \vec{T}(fy, gfy = hy)$ . Then  $h(x, y) = (c_{fx}, \tau) \circ f(x, y)$  and hence  $(c_{fx}, \tau) = h(x, y) \circ f(x, y)^{-1}$  is an isomorphism with inverse  $f(x, y) \circ h(x, y)^{-1}$ .

The pureness condition can be decided within the preorder category  $\vec{D}_\pi(X)$ : Let  $Inv(D_\pi(X))$  denote the pullback subcategory in the diagram

$$\begin{array}{ccc} & Aut(X)^{-1} & . \\ & \downarrow & \\ \vec{D}_\pi(X) & \longrightarrow & \vec{D}_\pi^{Aut}(X)[Aut(X)^{-1}] \end{array}$$

It is easy to check that

**Lemma 7.3.**  $(\vec{D}_\pi^{Aut}(X)[Aut(X)^{-1}], Aut(X)^{-1})$  is pure if and only if  $(\vec{D}_\pi(X), Inv(D_\pi(X)))$  is pure. If this is the case, isomorphisms in the component category can only split up into isomorphisms.

If the construction above does not result in a pure subcategory, the biggest subcategory of  $Aut(X)^{-1}$  satisfying SLEP/SREP will yield one, but there will in general be no control of its size.

The approach above will (for  $Aut(X)$ , not necessarily for  $Aut_{pm}(X)$ ) lead to particular components containing only elements in the diagonal as objects; both a future flow and a past flow preserve the diagonal.

**Example 7.4.** Using the approach described above yields almost the same component categories for the spaces  $X$  and  $\vec{S}^1$  from Example 4.6 as depicted previously.

**X:** Every future homotopy flow preserves  $A$ , since its maximum is a fixed point; likewise, every past homotopy flow preserves  $D$ . No flow line can ever connect a point in  $B$  to a point in  $C$ . For  $x \preceq y$ , the trace spaces  $\vec{T}(X)(x, y)$  are contractable unless  $x \in A$  and  $y \in B$ , in which case the trace space has two contractable components. It is not difficult to construct (piecewise linear) automorphic homotopy flows with flow lines connecting any two  $x \preceq y$  from a given "component". As a result, we get additional initial components  $\Delta_A, \Delta_B, \Delta_C, \Delta_D$ , that have to be taken from the components  $AA$  etc.

$\vec{S}^1$ : The component category has two objects: the diagonal  $\Delta$  in the torus  $S^1 \times S^1$  and its complement  $T'$ , the torus with the diagonal deleted. To see that pairs  $(\alpha_i, \beta_i), i = 1, 2$  are connected within  $Aut(X)$ , note that a rotation (in  $Aut(X)$ ) connects  $(\alpha_1, \beta_1)$  to  $(\alpha_2, \beta_1 + (\alpha_2 - \alpha_1))$ , which is connected to  $(\alpha_2, \beta_2)$  by an automorphism that inflates/deflates the arc starting at  $\alpha_2$  and its complement. The category has morphisms  $a : \Delta \rightarrow \Delta, b : T' \rightarrow T', c : \Delta \rightarrow T', d : T' \rightarrow \Delta$  with  $dc = a$  and  $cd = b$ .

*Remark 7.5.* The set-up goes through without major changes if one replaces automorphisms by automorphisms up to homology/homotopy in a given range of dimensions.

**7.4. Naturality issues.** As discussed earlier [12, 18, 7], a general  $d$ -map does in general *not* preserve components. Some coherence with the automorphic flows on the two spaces is needed.

**Definition 7.6.** (1) A  $d$ -map  $f : X \rightarrow Y$  is called *coherent* if, for every pair  $x \preceq y$  and every morphism  $F(x, y) \in Aut(X)((x, y), (Fx, Fy))$  there exists a morphism  $G \in Aut(Y)^{-1}((fx, fy)(fFx, fFy))$ .  
 (2)  $f$  is called *strongly coherent* if, for every strictly automorphic past/future homotopy flow  $H$  on  $X$ , there is a strictly automorphic homotopy flow  $\bar{H}$  on  $Y$  solving the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ H \uparrow & & \uparrow \bar{H} \\ X \times \vec{I} & \xrightarrow{f \times id_I} & Y \times \vec{I}. \end{array}$$

It is in general not easy to check for coherence unless one already knows the components of the two spaces. On the other hand, strong coherence (which applies coherence) is more rare, but it is, e.g., satisfied for certain inclusion maps. As a direct consequence of the definitions, we obtain

**Proposition 7.7.** A coherent  $d$ -map  $f : X \rightarrow Y$  satisfies  $\vec{T}^X(f)(Aut(X)) \subseteq Aut(Y)^{-1}$ . Hence it maps components of  $X$  into components of  $Y$  and induces a functor

$$f_{\#} : \pi_0(\vec{D}_{\pi}^{Aut}(X); Aut(X)) \rightarrow \pi_0(\vec{D}_{\pi}^{Aut}(Y); Aut(Y)).$$

□

## 8. COMPONENTS AND DIHOMOTOPY EQUIVALENCES

In the following, we collect what we know about the behaviour of dihomotopy equivalences on components. There are many examples of  $d$ -maps (e.g. , inclusions, cf. [12]) that do not respect components (with respect to whatever equivalence relation). It is not clear from the definitions either whether a dihomotopy equivalence always does; one needs to impose weak coherence (Definition 8.6) as an extra requirement; we will show that weakly coherent dihomotopy equivalences give rise to isomorphic component categories.

**8.1. Homotopy flows and components.** We start by considering the effects of strictly automorphic  $d$ -homotopies  $H_1 : id_X \rightarrow h_1$ , resp.  $H_2 : h_2 \rightarrow id_X$  on the components of a  $d$ -space  $X$  with respect to the subcategory  $Aut(X)$ . The following result follows immediately from the definitions:

**Lemma 8.1.** (1) *The pairs  $(h_2(x), h_2(y)) \mapsto (x, y) \mapsto (h_1(x), h_1(y))$  are contained in the same component for every object  $(x, y)$  in  $\vec{D}_\pi(X)$ .*  
 (2) *For  $h = h_i, i = 1, 2$ , the morphisms  $\tau \in \vec{D}_\pi(X)((x, y), (x', y'))$  and  $(h \circ \tau) \in \vec{D}_\pi(X)((hx, hy), (hx', hy'))$  are equivalent for every morphism  $\tau$  in the pre-order category.*  
 (3)  *$h$  induces the identity on the component category  $\pi_0(\vec{D}_\pi^{Aut}(X); Aut(X))$ .*

Analogous results hold for the other component categories considered in Section 7.

**8.2. Dihomotopy equivalences and components.** In order to phrase our first result concerning dihomotopy equivalences, we make use of

**Definition 8.2.** Let  $Y' \subset Y$  denote a (non-empty) subset of a  $d$ -space  $Y$ .  $Y'$  is called *dense* if for every pair  $(x, x')$  in  $\vec{D}(Y)$  there is a pair  $(y, y')$  in  $\vec{D}(Y')$  with  $(x, x') \preceq_D (y, y')$ .

*Remark 8.3.* The definition corresponds in fact to denseness in an appropriate order topology.

In the following, we consider components with respect to  $Aut(X) \subseteq \vec{D}_\pi^{Aut}(X)$ .

**Proposition 8.4.** *Let  $f : X \rightarrow Y$  denote a dihomotopy equivalence. Then*

- (1) *The intersection of the image  $f(\vec{D}_\pi(X)) \subset \vec{D}_\pi(Y)$  with every component in  $\vec{D}_\pi(Y)$  is dense.*
- (2) *If  $f(x_1) = f(x_2), f(x'_1) = f(x'_2), x_i, x'_i \in X$ , then  $(x_1, x'_1)$  and  $(x_2, x'_2)$  are contained in the same component of  $\vec{D}_\pi(X)$ .*
- (3) *Every morphism  $\beta$  in  $\vec{D}_\pi(Y)$  is  $Aut(Y)$ -equivalent to a morphism  $f \circ \alpha$  with  $\alpha$  a morphism in  $\vec{D}_\pi(X)$ .*
- (4) *If  $f \circ \alpha_1 = f \circ \alpha_2$  for morphisms  $\alpha_i$  in  $\vec{D}_\pi(X)$ , then  $\alpha_1$  and  $\alpha_2$  are  $Aut(X)$ -equivalent.*

Similar results hold also for the other component categories considered previously.

*Proof.* Let  $g : Y \rightarrow X$  denote a dihomotopy inverse to  $f$ .

- (1) By Lemma 8.1(1),  $(y, y')$  and  $(fgy, fgy')$  are contained in the same component in  $\vec{D}(Y)$ .
- (2)  $(x_i, x'_i)$  is contained in the same component as  $(gfx_1, gfx'_1) = (gfx_2, gfx'_2)$ .
- (3)  $\beta$  is  $\text{Aut}(Y)$ -equivalent to  $fg(\beta)$ .
- (4)  $\alpha_i$  is  $\text{Aut}(X)$ -equivalent to  $gf(\alpha_1) = gf(\alpha_2)$ .

□

Dihomotopy equivalences behave well with respect to preserving the topology of trace spaces:

**Proposition 8.5.** *Let  $f : X \rightarrow Y$  denote a dihomotopy equivalence. Let  $(x, y), (x', y')$  be objects of  $D_\pi(X)$  such that  $\vec{T}(x, y)$  and  $\vec{T}(x', y')$  are homotopy equivalent. Then the trace spaces  $\vec{T}(fx, fy)$  and  $\vec{T}(fx', fy')$  are homotopy equivalent, too. Similarly for trace spaces that induce isomorphisms in homology or homotopy in a range of dimensions.*

*Proof.* The diagram

$$\begin{array}{ccccc}
 \vec{T}(Y)(fx, fy) & \xrightarrow[\simeq]{\vec{T}(g)} & \vec{T}(X)(gfx, gfy) & \xleftarrow[\simeq]{\vec{T}(gf)} & \vec{T}(X)(x, y) \\
 & & & & \downarrow \simeq \\
 \vec{T}(Y)(fx', fy') & \xrightarrow[\vec{T}(g)]{\simeq} & \vec{T}(X)(gfx', gfy') & \xleftarrow[\vec{T}(gf)]{\simeq} & \vec{T}(X)(x', y')
 \end{array}$$

yields the homotopy equivalence asked for.

□

### 8.3. Weakly coherent dihomotopy equivalences.

**8.3.1. Definition.** Unfortunately, Proposition 8.5 does not imply that a dihomotopy equivalence induces isomorphisms of component categories. In order to get functoriality, a coherence condition has to be imposed. Note that the condition below is weaker than asking the map  $f$  to be coherent itself in the sense of Definition 7.6.

**Definition 8.6.** Let  $f : X \rightarrow Y$  denote a (future/past) dihomotopy equivalence with homotopy inverse  $g : Y \rightarrow X$ . The pair  $(f, g)$  is called *weakly coherent*, if the  $d$ -self maps  $g \circ f : X \rightarrow X$  and  $f \circ g : Y \rightarrow Y$  are coherent (cf. Definition 7.6). The map  $f$  is called *weakly coherent* itself, if there exists a homotopy inverse such that  $(f, g)$  is weakly coherent.

The extra coherence requirement will be crucial in the investigation of the effect of a dihomotopy equivalence from  $X$  to  $Y$  on component categories associated to these two  $d$ -spaces. One may ask how natural weak coherence is:

- Remark 8.7.*
- (1) In an analogous situation, a self-diffeomorphism  $F : X \rightarrow X$  conjugates a flow diffeomorphism  $h_1 : X \rightarrow X$  to the flow diffeomorphism  $h_2 = F \circ h_1 \circ F^{-1}$  – of a different dynamical system.
  - (2) The coherence condition corresponds to a weaker form of the coherence requirement in the definition of future and past equivalences in [16].

### 8.3.2. Coherent dihomotopy equivalences induce isomorphisms of component categories.

**Proposition 8.8.** *A weakly coherent dihomotopy equivalence  $f : X \rightarrow Y$  induces an isomorphism  $\pi_0 \vec{T}(f) : \pi_0(\vec{D}_\pi(X), \text{Aut}(X)) \rightarrow \pi_0(\vec{D}_\pi(Y), \text{Aut}(Y))$  of component categories.*

*Proof.* The only substantial difficulty arises in proving the *existence* of a functor  $\pi_0 \vec{T}(f)$  as above; it is here that weak coherence is needed. Let  $h_1 \in \text{Aut}_+(X)$  and  $(x, y) \in \vec{D}_\pi(X)$ . We have to show, that there is an  $\text{Aut}(Y)^{-1}$ -morphism connecting  $(fx, fy)$  and  $fh_1x, fh_1y$ .

Since  $gf : X \rightarrow X$  is coherent, there exists  $h_2 \in \text{Aut}(X)^{-1}$  with  $(gh_1x, gh_1y) = (h_2gh_1x, h_2gh_1y)$ . Then  $fh_2g : Y \rightarrow Y$  is an automorphism (see the proof of Proposition 6.6) that connects  $(fx, fy)$  to  $(fh_2gh_1x, fh_2gh_1y)$ , whereas the automorphism  $fg$  connects  $(fh_1x, fh_1y)$  to  $(fgfx, fgfh_1y) = (fgfx, fh_2gh_1y)$ :

$$\begin{array}{ccc} (fh_2gh_1x, fh_2gh_1y) & \xleftarrow{=} & (fgfx, fh_2gh_1y) \\ \uparrow fh_2g(x,y) & & \uparrow fg(fh_1x, fh_1y) \\ (fx, fy) & & (fh_1x, fh_1y) \end{array}$$

Iterating the argument above, (for a zig-zag path in  $\vec{D}_\pi^{\text{Aut}}(X)$ ) shows that equivalent pairs in  $\vec{D}_\pi^{\text{Aut}}(X)$  are mapped to equivalent pairs in  $\vec{D}_\pi^{\text{Aut}}(Y)$  under  $\vec{T}_\pi(f)$ . The construction is well-behaved with respect to concatenation maps and yields therefore a functor  $\pi_0 \vec{T}(f) : \pi_0(\vec{D}_\pi^{\text{Aut}}(X), \text{Aut}(X)) \rightarrow \pi_0(\vec{D}_\pi^{\text{Aut}}(Y), \text{Aut}(Y))$

Using the coherence of  $fg : Y \rightarrow Y$ , one obtains a reverse functor  $\pi_0 \vec{T}(g) : \pi_0 \vec{D}_\pi(Y) \rightarrow \pi_0 \vec{D}_\pi(X)$ . By Lemma 8.1(3), the compositions of the two functors yield identity functors on the component categories of  $X$ , resp.  $Y$ .

□

In conclusion, if two  $d$ -spaces have non-isomorphic component categories, then there cannot exist a weakly coherent dihomotopy equivalence between them.

**8.4. A more general perspective?** The preceding sections dealt only with categories arising from path spaces of (topological)  $d$ -spaces as quotients etc. A straightforward generalization of the approach to more general categories can be described as follows:

We consider small categories *over* a given small category  $\mathcal{D}$ , i.e., every category  $\mathcal{C}$  is endowed with a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  into  $\mathcal{D}$ ; typically,  $\mathcal{D} = \text{Ho} - \text{Top}, \text{Ab}$  etc.

Given a category  $F : \mathcal{C} \rightarrow \mathcal{D}$  over  $\mathcal{D}$ , consider an endofunctor  $\Phi : \mathcal{C} \rightarrow \mathcal{C}$  together with a “directed homotopy”  $\varphi : 1_{\mathcal{C}} \rightarrow \Phi$  in the sense of [14], i.e., a natural transformation. Such a pair  $(\Phi, \varphi)$  is called a *future auto-equivalence over  $\mathcal{D}$*  if the morphisms  $F(\varphi(x)) : F(x) \rightarrow F(\Phi(x))$  are  $\mathcal{D}$ -isomorphisms for *all* objects  $x$  in  $\mathcal{C}$ . Note the following consequence for every  $\mathcal{C}$ -morphism  $a$ :  $F(a)$  is a  $\mathcal{D}$ -isomorphism if and only if  $F(\Phi(a))$  is a  $\mathcal{D}$ -isomorphism. Past auto-equivalences are defined in the same way using directed homotopies  $\varphi : \Phi \rightarrow 1_{\mathcal{C}}$ .

Let  $M_{\mathcal{C}}$  denote a *monoid* of such auto-equivalences over  $\mathcal{D}$ . Two  $\mathcal{C}$ -objects are then  $M_{\mathcal{C}}$ -equivalent if there is a zig-zag of  $M_{\mathcal{C}}$ -auto-equivalences relating them.

Two functors  $F_i : \mathcal{C}_i \rightarrow \mathcal{D}$  can be related by a functor pair  $(\Psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2, \psi : \mathcal{D} \rightarrow \mathcal{D})$  such that the diagram

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Psi} & \mathcal{C}_2 \\ F_1 \downarrow & & \downarrow F_2 \\ \mathcal{D} & \xrightarrow{\psi} & \mathcal{D} \end{array}$$

commutes on both objects and morphisms.

Let  $M_{\mathcal{C}_1}$ , resp.  $M_{\mathcal{C}_2}$  denote monoids of auto-equivalences over  $\mathcal{D}$ . A functor pair  $(\Psi, \psi) : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  over  $\mathcal{D}$  is then called a *future homotopy equivalence* if there is a functor pair  $(\Gamma, \gamma) : \mathcal{C}_2 \rightarrow \mathcal{C}_1$  over  $\mathcal{D}$  such that every  $\mathcal{C}$ -object  $x$  is  $M_{\mathcal{C}_1}$ -equivalent to  $(\Gamma \circ \Psi)(x)$ , and every  $\mathcal{C}_2$ -object  $y$  is  $M_{\mathcal{C}_2}$ -equivalent to  $(\Psi \circ \Gamma)(y)$ .

Arguing formally as in the proof of Proposition 6.4, it can then be seen that  $\psi : F_1(x) \rightarrow F_2(\Psi(x))$  is an isomorphism for every object  $x$  in  $\mathcal{C}_1$ .

**Example 8.9.** Compare Example 7.4. Let  $\mathcal{D}$  denote the category with one object representing the monoid  $\mathbf{N}_{\geq 0} = \{0, 1, 2, \dots\}$  of non-negative integers; its morphisms correspond to the  $k$ -th successor functions on  $\mathbf{N}_{\geq 0}$ ,  $k \geq 0$ ; the only invertible morphism corresponds to 0.

Let  $\mathcal{C}_1$  denote the preorder category of the circle  $\bar{S}^1$ . With  $\exp : \mathbf{R} \rightarrow S^1$  denoting the exponential function  $\exp(t) = e^{2\pi it}$ , we consider the following functor  $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}$ : Let  $\omega$  denote a  $d$ -path on  $S^1$  from  $y$  to  $y'$  considered as a morphism in  $\mathcal{D}(\bar{S}^1)((x, y), (x, y'))$ . Let  $\alpha$  denote a shortest  $d$ -path from  $x$  to  $y$ , and consider any lift  $l(\alpha * \omega)$  from  $x$  to  $y'$ . Let  $F_1(\omega) = \lfloor l(\alpha * \omega) \rfloor$ , the integral part of the length (winding number) of that lift. The morphism  $\omega$  is thus  $\mathcal{D}$ -invertible if  $|l(\alpha * \omega)| < 1$ .  $F_1$  is in fact functorial, whereas a similar construction does *not* work if the preorder category is replaced by the fundamental category.

A  $d$ -map  $f : S^1 \rightarrow S^1$  induces an auto-equivalence of  $\mathcal{C}_1$  over  $\mathcal{D}$  if and only if it is of degree one (ensuring the existence of  $d$ -homotopies to and from  $id_{S^1}$ ) and if it is *injective*: If  $fx = fy$  for some  $x \neq y$ , i.e,  $f$  is constant on the arc from  $x$  to  $y$ , then any  $d$ -path  $\omega$  from  $y$  to  $x$  (of length less than 1) is mapped to a loop (of length 1). Hence one may choose for  $M_{\mathcal{C}_1}$  the transformations induced by injective degree one self  $d$ -maps. Two equivalence classes arise on the objects of  $\mathcal{C}_1$ : the diagonal  $\Delta$  and its complement  $\bar{\Delta}$ .

Let  $\mathcal{C}_2$  denote the category on two objects  $\Delta, \bar{\Delta}$  freely generated by two morphisms  $a \in \mathcal{C}_2(\Delta, \bar{\Delta})$  and  $b \in \mathcal{C}_2(\bar{\Delta}, \Delta)$ . Let  $F_2 : \mathcal{C}_2 \rightarrow \mathcal{D}$  denote the functor determined by  $F_2(a) = 0$  and  $F_2(b) = 1$ . The only auto-equivalence of this category over  $\mathcal{D}$  is the identity.

A functor  $\Psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  over the identity is given by

$$\Psi(x, y) = \begin{cases} \Delta & x = y, \\ \bar{\Delta} & x \neq y \end{cases},$$

and  $\Psi(\omega)$  is the morphism between the corresponding objects that satisfies:  $F_1(\omega) = F_2(\Psi(\omega))$ . A reverse functor  $\Gamma : \mathcal{C}_2 \rightarrow \mathcal{C}_1$  over the identity is defined by  $\Gamma(\Delta) = (1, 1), \Gamma(\bar{\Delta}) = (1, -1)$ ;  $\Gamma(a)$  is represented by the arc from 1 to  $-1$  and  $\Gamma(b)$  by the arc from  $-1$  to 1. Obviously,  $\Psi \circ \Gamma = id$  whereas  $\Gamma \circ \Psi$  preserves the diagonal and its complement.

There is good reason to distinguish the diagonal and its complement: The only isomorphism with target  $(x, x)$  is the identity, whereas there are infinitely many objects  $(x, y')$  with isomorphisms into  $(x, y)$  for  $x \neq y$ .

## REFERENCES

1. H.J. Baues, *Algebraic homotopy*, Cambridge Studies in Advanced Mathematics, no. 15, Cambridge University Press, 1989.
2. M. Bednarczyk, A. Borzyskowski, and W. Pawłowski, *Generalized congruences - epimorphisms in Cat*, Theory Appl. Categ. **5** (1999), no. 11, 266–280.
3. T. Borceux, *Handbook of Categorical Algebra I: Basic Category Theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1994.
4. G. Carlsson, A. Zomorodian, A. Collins, and L. Guibas, *Persistence Barcodes for Shapes*, Int. J. Shape Modelling **11** (2005), 149–187.
5. U. Fahrenberg and M. Raussen, *Reparametrizations of continuous paths*, Tech. Report R-2006-22, Department of Mathematical Sciences, Aalborg University, 2006.
6. L. Fajstrup, *Dipaths and Dihomotopies in a cubical complex*, Adv. in Appl. Math **35** (2005), no. 2, 188–206.
7. L. Fajstrup, É. Goubault, E. Haucourt, and M. Raussen, *Components of the fundamental category*, Appl. Categ. Structures **12** (2004), no. 1, 81–108.
8. L. Fajstrup, É. Goubault, and M. Raussen, *Algebraic topology and concurrency*, Theor. Comput. Sci **357** (2006), 241–278.
9. P. Gaucher, *A model category for the homotopy theory of concurrency*, Homology, Homotopy Appl. **5** (2003), no. 2, 549–599.
10. ———, *Inverting weak dihomotopy equivalence using homotopy continuous flow*, Theory Appl. Categ. **16** (2006), no. 3, 59–83.
11. É. Goubault and E. Haucourt, *Components of the Fundamental Category II*, Tech. report, CEA, 2006.
12. É. Goubault and M. Raussen, *Dihomotopy as a tool in state space analysis*, LATIN 2002: Theoretical Informatics (Cancun, Mexico) (S. Rajsbaum, ed.), Lect. Notes Comput. Sci., vol. 2286, Springer-Verlag, April 2002, pp. 16 – 37.
13. M. Grandis, *Directed Homotopy Theory II. Homotopy Constructs*, Theory Appl. Categ. **10** (2002), no. 14, 369 – 391.
14. ———, *Directed Homotopy Theory I. The Fundamental Category*, Cahiers Topologie Géom. Différentielle Catégoriques **44** (2003), 281 – 316.
15. ———, *Modelling fundamental 2-categories for directed homotopy*, Tech. report, Dip. Mat. Univ. Genova, 2005.
16. ———, *The shape of a category up to directed homotopy*, Theory Appl. Categ. **15** (2005), no. 4, 95 – 146.
17. S. Krishnan, *A convenient category of locally ordered spaces*, GETCO '05, Preliminary Proceedings (San Francisco), vol. NS-05, BRICS, 2005, full version to appear in ENTCS Proceedings, p. 33.
18. M. Raussen, *State spaces and dipaths up to dihomotopy*, Homology, Homotopy Appl. **5** (2003), no. 2, 257–280.
19. T. tom Dieck, *Transformation Groups*, Studies in Mathematics, vol. 8, de Gruyter, Berlin, New York, 1987.

20. R.J. van Glabbeek, *On the Expressiveness of Higher Dimensional Automata*, Express 2004, 2005, Proceedings of the 11th International Workshop on Expressiveness in Concurrency, pp. 5–34.
21. A. Zomorodian, *Topology for computing*, Cambridge monographs on applied and computational mathematics, vol. 16, Cambridge Univ. Press, 2005.

DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, DK - 9220 AALBORG ØST

*E-mail address:* raussen@math.aau.dk